WPI Department of Mathematical Sciences 503 GCE
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Name:______________________________

Exercise 1:
Let (X, ρ) be a metric space and S and T two non-empty subsets of X. Define

$$d(S,T) = \max\{\sup_{x \in S} \inf_{y \in T} \rho(x,y), \sup_{y \in T} \inf_{x \in S} \rho(x,y)\}.$$

Show that $d(S,T) = 0$ if and only if S and T have the same closure.

Exercise 2:
Show that for every set $S \subset \mathbb{R}$ there exists a Borel set B such that $S \subset B$ and $m^*(S) = m^*(B)$, where m^* is the Lebesgue outer measure. Then show that for such S and B with $m^*(S) < \infty$, S is measurable if and only if $m^*(B \setminus S) = 0$.

Exercise 3:
Suppose f_n, g_n are Lebesgue measurable functions on $\mathbb{R}$, with $f_n, g_n \geq 0 \ \forall n \in \mathbb{N}$. Suppose also that $f_n \to f$ a.e., $g_n \to g$ a.e.,

$$\int f_n \to \int f < \infty,$$

and

$$\int g_n \to \int g < \infty.$$

Prove or give a counterexample: if $\{f_n g_n\}$ is bounded in L^1, then

$$\int f_n g_n \to \int fg.$$