WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 541 Probability and Mathematical Statistics II

August, 2020

Note: The problems are arbitrarily ordered, not necessarily according to difficulty. Please show a clear logic of your solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) The shifted Exponential: Let $X_1, \cdots, X_n$ be a random sample from the density
$$f(x|\theta, \alpha) = \frac{1}{\theta}\exp(-\frac{x-\alpha}{\theta})I(\alpha, \infty)(x),$$
for $\theta > 0$ and real α. Suppose θ and α are unknown.
   (a) Derive the MLE for (θ, α). Hint: First find the MLE for α, then use it to find the MLE for θ.
   (b) Derive the method of moment estimator for (θ, α).

2. (20 points) Let $X_1, X_2 \ldots, X_n$ be *iid* with one of two probability density functions. If $\theta = 0$, then
$$f(x|\theta) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$
and if $\theta = 1$, then
$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}.$$
Find the maximum likelihood estimator of θ.

3. (20 points) Let $\{\varepsilon_j, j = 1, 2, \ldots\}$, $\{X_j, j = 1, 2, \ldots\}$, and $\{Y_j, j = 1, 2, \ldots\}$ be three independent sequences of iid random variables with
$$X_j \sim N(\mu_1, 1), \; Y_j \sim N(\mu_2, 1), \; \mathrm{P}(\varepsilon_j = 0) = \mathrm{P}(\varepsilon_j = 1) = 1/2,$$
where μ_1 and μ_2 are unknown parameters, and $\mu_1 > \mu_2$. Suppose that the mixture $W_j = \varepsilon_j X_j + (1 - \varepsilon_j)Y_j$ is observed for $j = 1, \ldots, n$, but ε_j, X_j, and Y_j, $j = 1, \ldots, n$ can not be observed.
   1. Find $\mathrm{E}(W_j)$ and $\mathrm{E}(W_j^2)$.
   2. Construct consistent estimators for μ_1 and μ_2 respectively based on $\{W_1, \ldots, W_n\}$. (Hint: Use the method of moments.)

4. (20 points) Consider testing simple hypotheses $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ based on a random sample with continuous distribution. The likelihood function satisfies $L(\theta_1|x) > L(\theta_0|x)$, where $x = (x_1, \cdots, x_n)$.

(a) Give the general definitions of the power function, the size-α test, and the uniformly most powerful (UMP) size-α test, and their meanings in the context of above hypothesis testing.

(b) In the context of above hypothesis testing, prove that the likelihood ratio test with size α is a UMP size-α test.

(c) In the context of above hypothesis testing, prove that any UMP size-α test is equivalent to the likelihood ratio test almost surely (when they exist).

5. (20 points) Let $X, Y \overset{ind}{\sim} \text{Gamma}(1, \theta)$ and suppose a scientist uses $T = X - Y$ to make inference about θ. What is the shortest $100(1-\alpha)\%$ confidence interval the scientist can obtain? What advice can you give the scientist in terms of a better confidence interval? Give your solution. $[f(x) = \theta e^{-\theta x}, x \geq 0.]$

6. (20 points) Let $X_1, \ldots, X_n \mid \mu \overset{ind}{\sim} \text{Normal}(\mu, \sigma^2)$, where σ^2 is known. Find $\text{E}\{\Phi(\bar{X})\}$, where $\bar{X}$ is the sample mean. What optimality property does the estimator, $\Phi(\bar{X})$, have? Find an approximate value of $\text{Var}\{\Phi(\bar{X})\}$ and say how you might estimate it approximately.