Mathematics in Computer Graphics and Games

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About Me

- Professor in WPI Computer Science Dept
- Grad school at Umass Amherst (MS, PhD)
  - Research in Computer graphics for 20 years
  - Taught computer graphics for 16 years

What is Computer Graphics (CG)?

- Computer graphics: algorithms, mathematics, programs ..... that **computer uses to generate PRETTY PICTURES**
- E.g Techniques to draw a line, polygon, cube

Computer-Generated!
Not a picture!

Uses of Computer Graphics

- **Entertainment:** games

Courtesy: Final Fantasy XIV

Courtesy: Super Mario Galaxy 2

Uses of Computer Graphics

- movies, TV (special effects, animated characters)

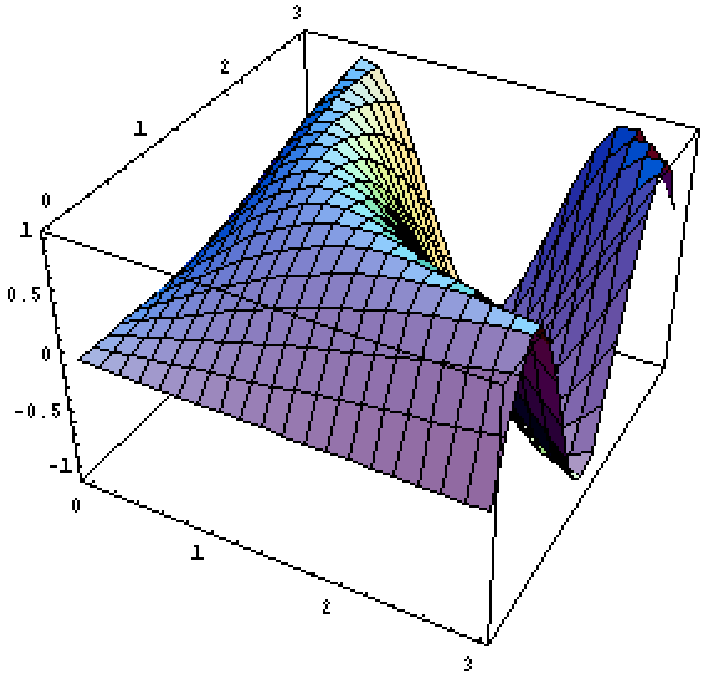
Courtesy: Shrek

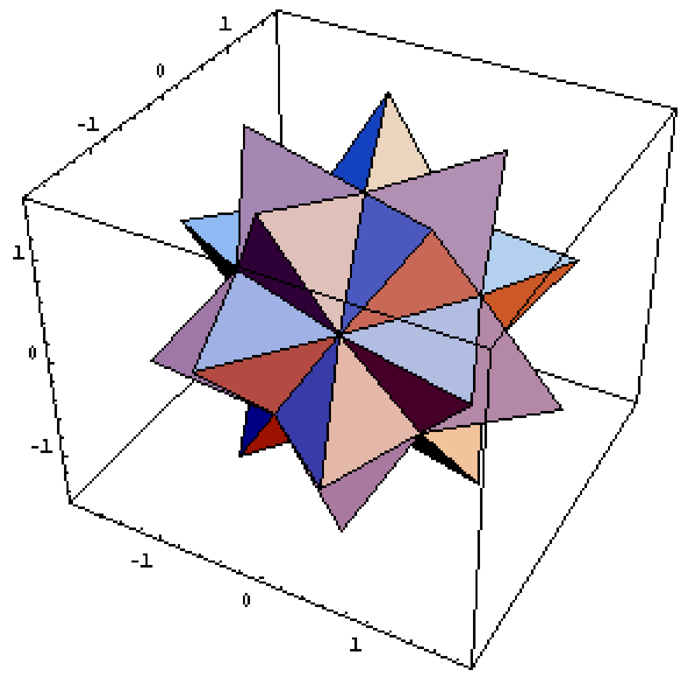
Courtesy: Spiderman

Note: Games and Movie industries Are two biggest hirers of computer Graphics professionals!!

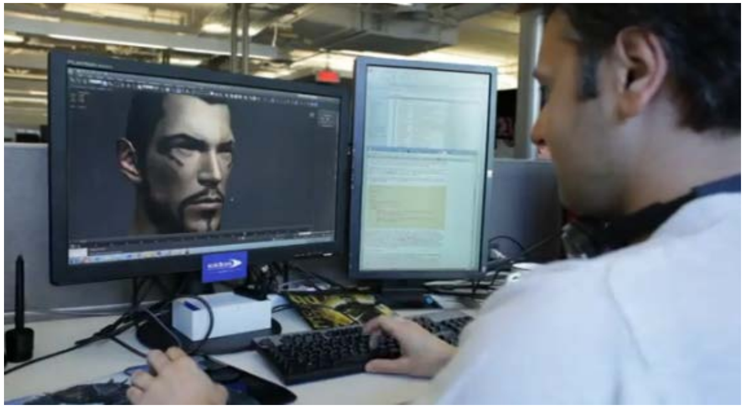
Uses of Computer Graphics

- Displaying Mathematical Functions
  - E.g., Mathematica®





2 Main Career Paths in Computer Graphics



1. Artist: Designs characters

Clicking, dragging, artsy!
No math skills required!!

2. Programmer: Writes programs to Make characters move, talk, etc

Math, programming skills required!!

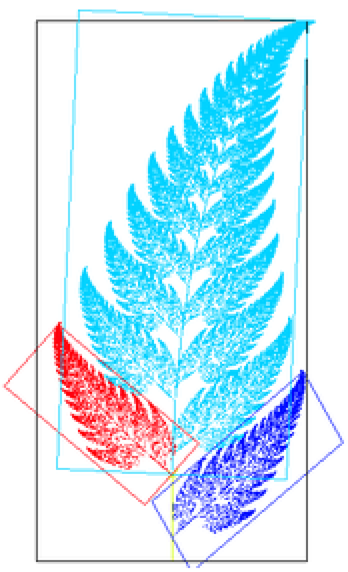
Your students probably Follow programmer path

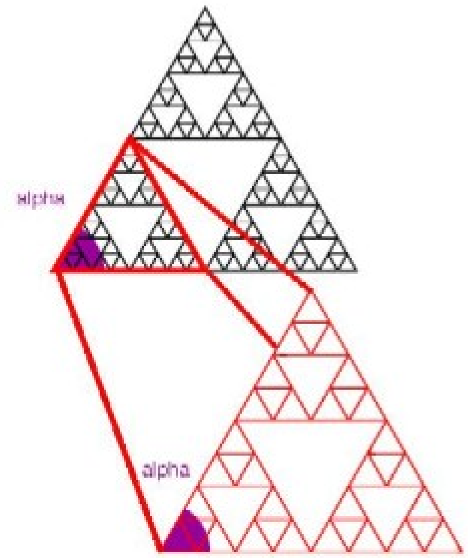
Some High School Math Used in CG

- Geometry
- Linear algebra: Matrices, vectors
- Trigonometry
- Complex numbers
- Boolean logic
- Probability

Fractals

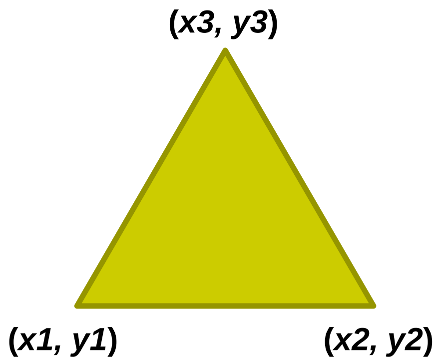
- Mathematical expressions to generate pretty pictures
- Evaluate math functions to create drawings
  - Evaluated function approached infinity -> converge to image
  - i.e f(1), f(2), f(3)…….. F(∞)
- Fractal image exhibits self-similarity: See similar sub-images within image as we zoom in



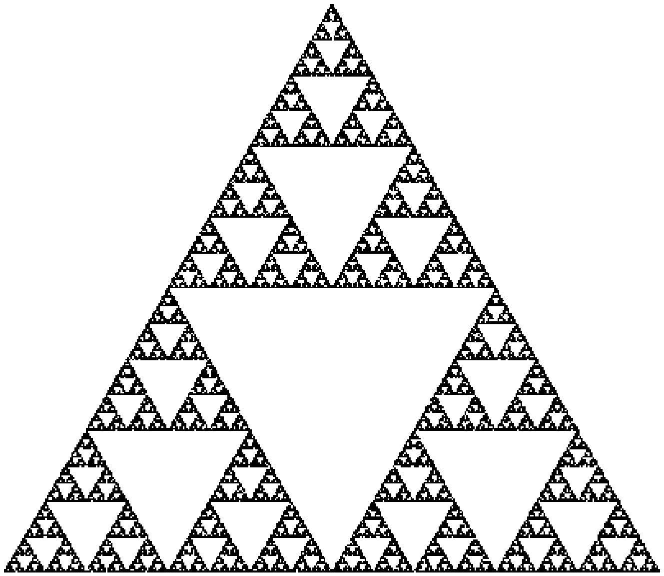


Sierpinski Gasket: Popular Fractal

Start with initial triangle with corners

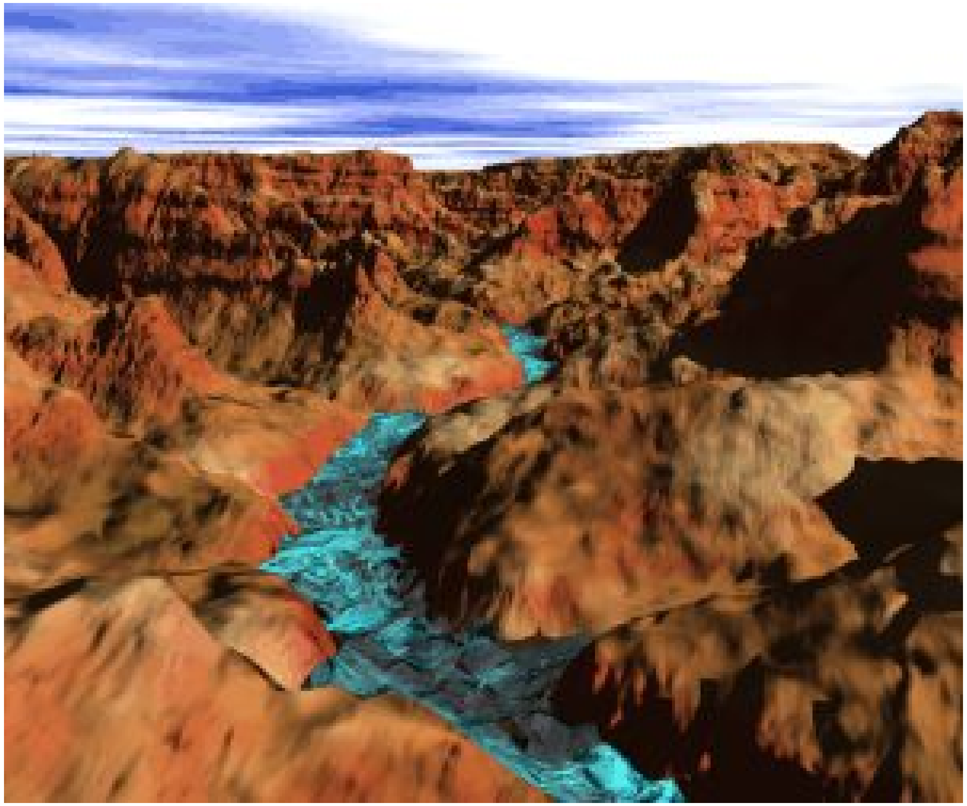


1. Pick initial point **p** = (*x*, *y*) at random inside triangle
2. Randomly select 1 of 3 vertices
3. Find **q,** halfway between **p** and randomly selected vertex
4. Draw dot at **q**
5. Replace **p** with **q**
6. Return to step 2

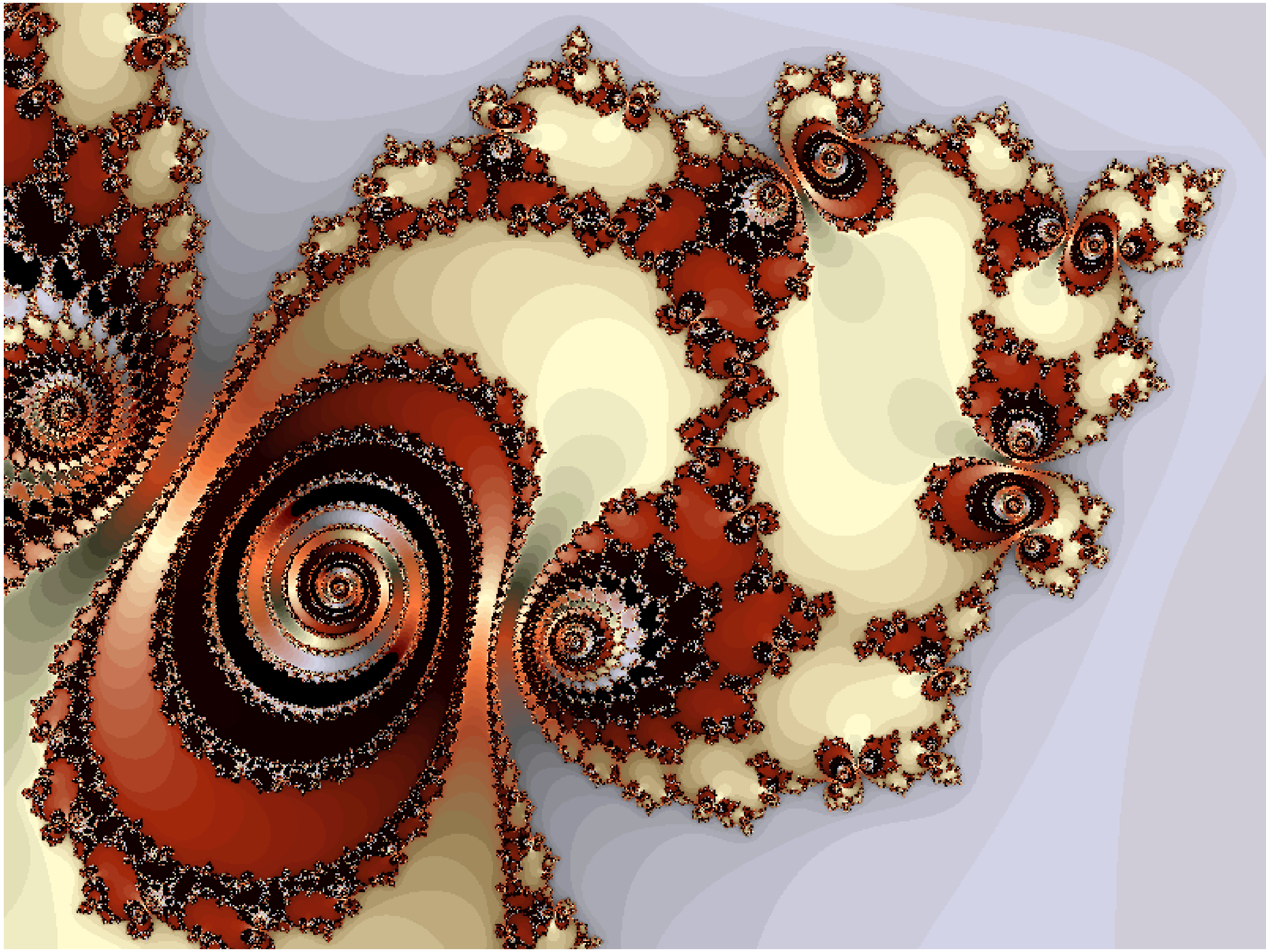


Example: Fractal Terrain

Terrain designed with only fractals

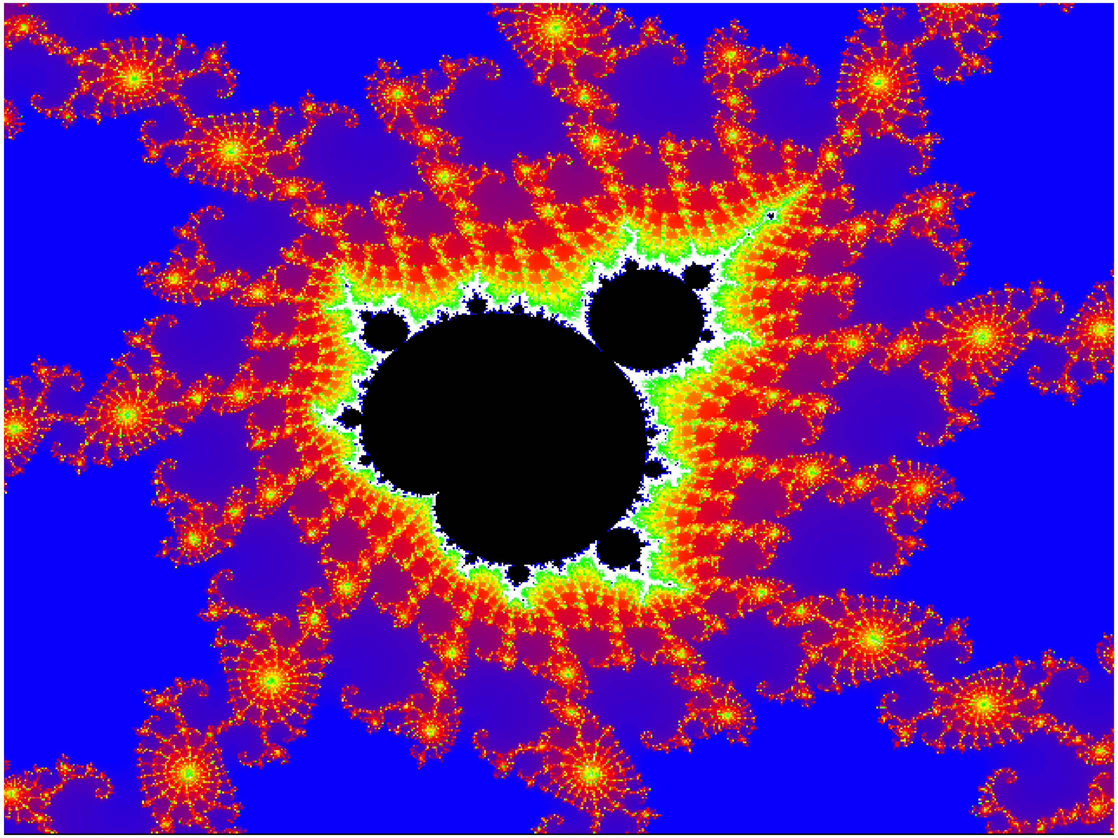


Example: Fractal Art



Courtesy: Internet Fractal Art Contest

Example: Mandelbrot Set



Mandelbrot Set

- Function of interest:

$$f(z) = (s)^2 + c$$

- Pick constants *s* and *c*
- **Orbit:** sequence of values (i.e d_1, d_2, d_3, d_4, *etc*):

$$d_1 = (s)^2 + c$$

$$d_2 = ((s)^2 + c)^2 + c$$

$$d_3 = (((s)^2 + c)^2 + c)^2 + c$$

$$d_4 = ((((s)^2 + c)^2 + c)^2 + c)^2 + c$$

- Question: does the orbit converge to a value?

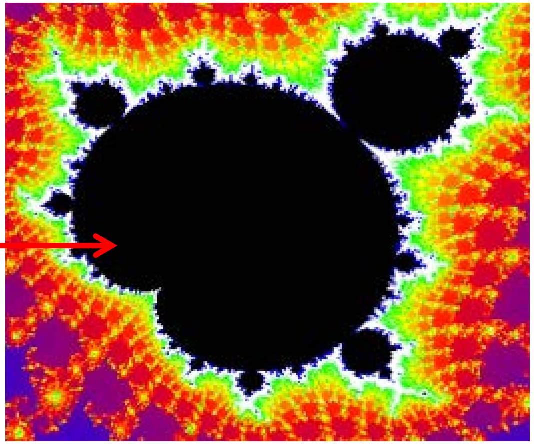
Mandelbrot Set

- Examples orbits:
  - $s = 0$, $c = -1$, orbit = 0,-1,0,-1,0,-1,0,-1,.....*finite*
  - $s = 0$, $c = 1$, orbit = 0,1,2,5,26,677...... *explodes*
- Orbit depends on *s* and *c*
- Basic question:
  - For given *s* and *c*,
    - does function stay finite? (within Mandelbrot set)
    - explode to infinity? (outside Mandelbrot set)

- Definition: if |d| < 2, for 1st 100 terms, orbit is finite else inifinite

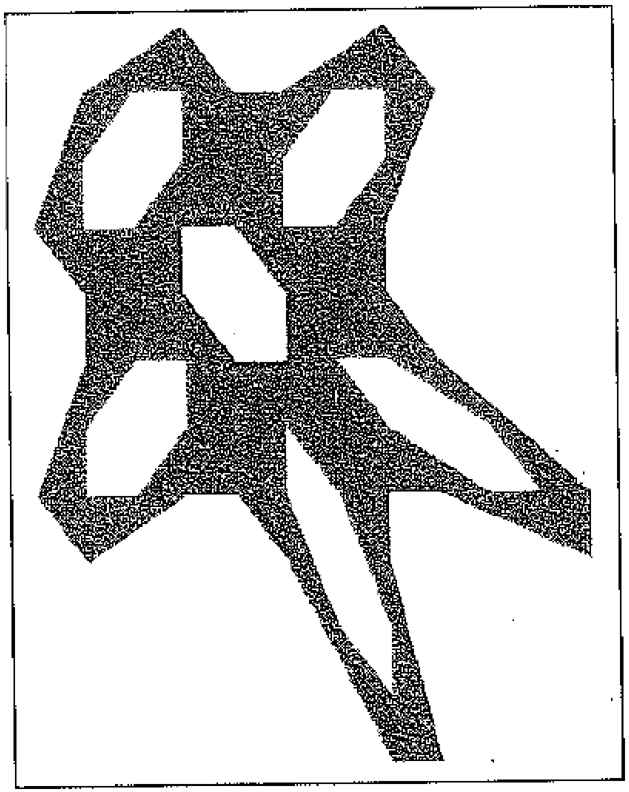
Mandelbrot Set

- Mandelbrot set: use complex numbers for *c* and *s*
- Set *s* = 0, *c* as a complex number
- E.g: *s* = 0, *c* = 0.2 + 0.5i
- Definition: Mandelbrot set includes all finite orbit *c*
- Mandelbrot set program:
  - Choose s and c,
  - program calculates d_1, d_2, d_3, d_4 and tests if they are finite
  - Choose colors



Values of c in mandelbrot set

Other Fractal Examples



Gingerbread Man

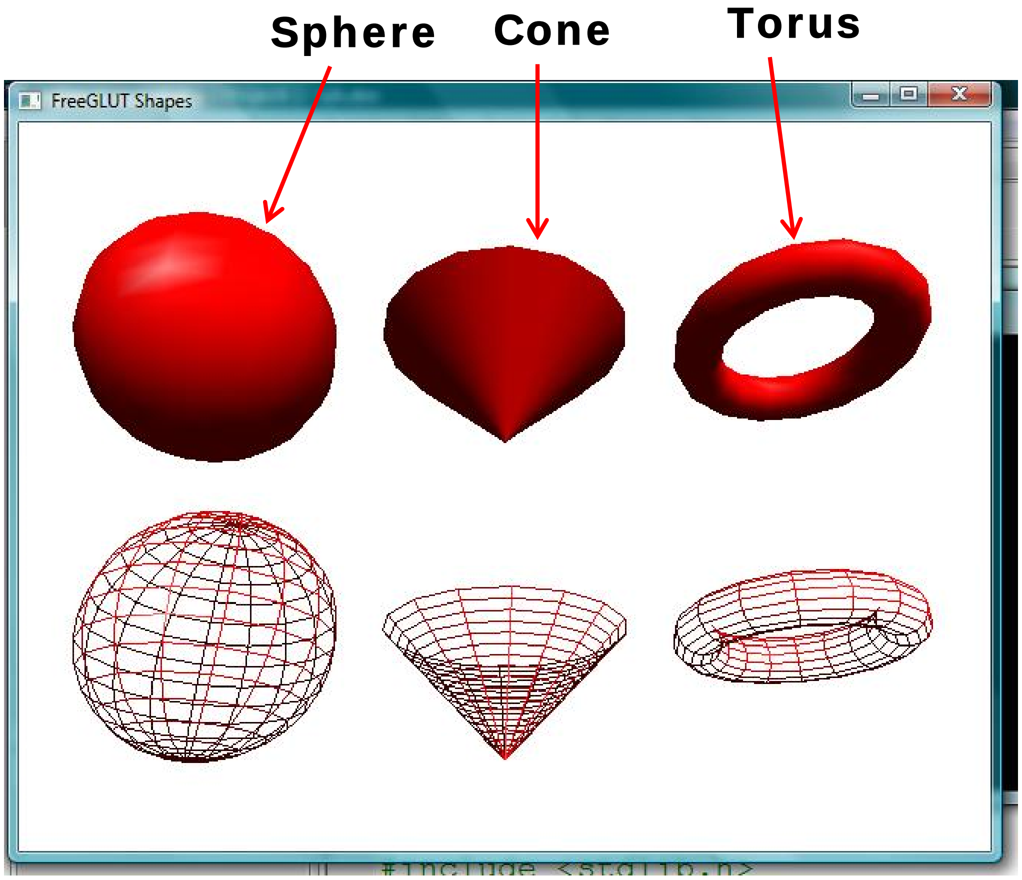
The Fern

Geometric Representations: 3D Shapes

- Can generate shapes using closed form geometric equations
- Example: Sphere

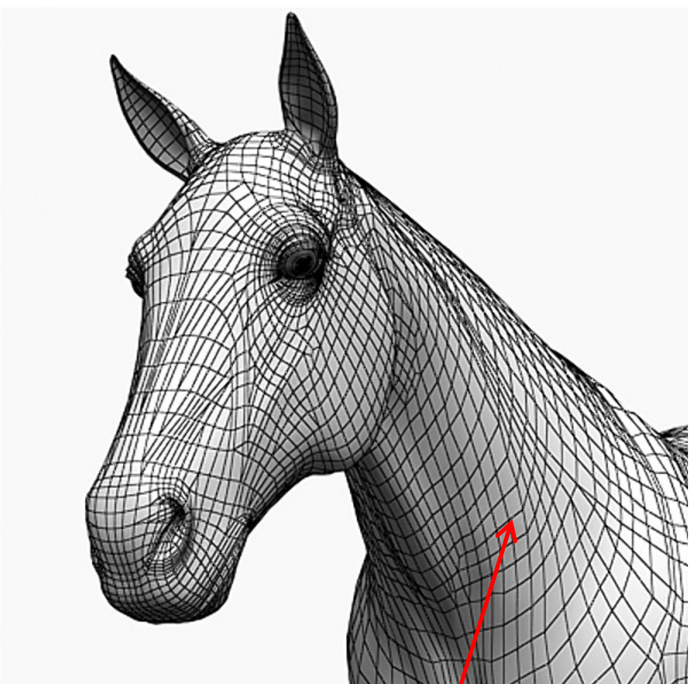
$$x^2 + y^2 + z^2 = R^2,$$

- **Problem:** A bit restrictive to design real world scenes made of spheres, cones, etc



Geometric Representations: Meshes

- Collection of polygons, or faces, that form “skin” of object
- More flexible, represents complex surfaces better
- Mesh? List of (x,y,z) points + connectivity
- Digitize real objects: very fine mesh

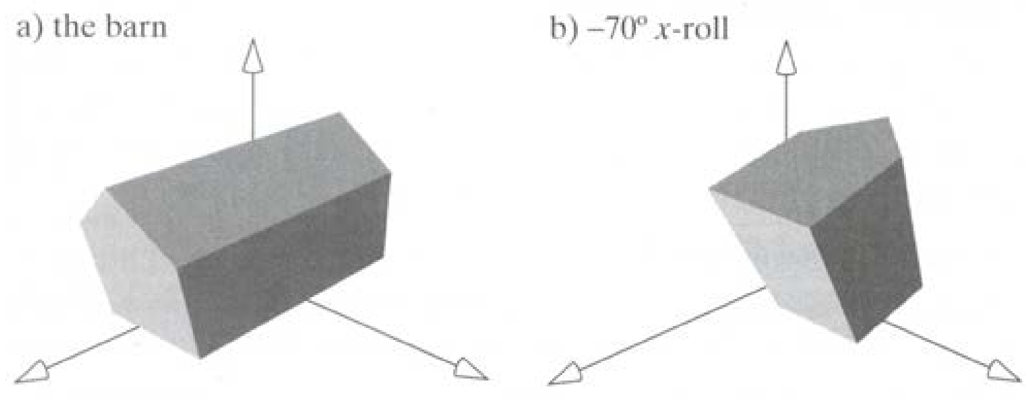


Each face of mesh is a polygon

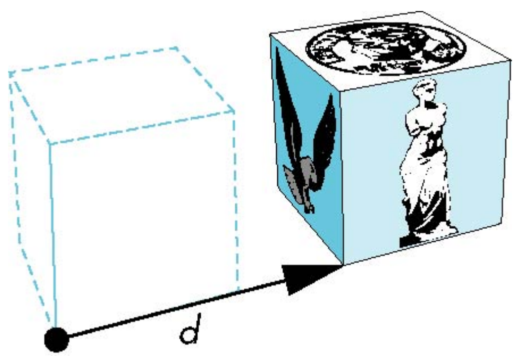
Digitized mesh of statue of Lucy: 28 million faces

Affine Transformations

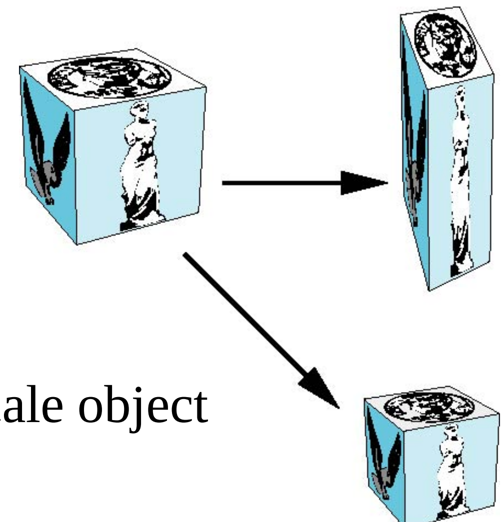
- Translation
- Scaling
- Rotation
- Shear



Rotate object



Translate object



Scale object

Affine Transforms: General Approach

- We can transform (translation, scaling, rotation, shearing, etc) object by applying matrix multiplications to object vertices

$$\begin{pmatrix} P_x' \\ P_y' \\ P_z' \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Transformed Vertex

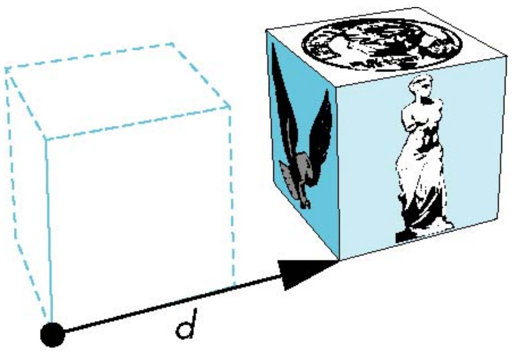
Transform Matrix
(e.g. rotate, translate, scale, etc)

Original Vertex

- Note: point (x,y,z) needs to be represented as (x,y,z,1), also called **Homogeneous coordinates**

3D Translation using Matrices

- Move each object vertex by same distance **d = (d_x, d_y, d_z)**
- **Example:** If we translate a point (2,2,2) by displacement (2,4,6), new location of point is (4,6,8)



Translate object

(2,2,2) + (2,4,6) = (4,6,8)

- Translate x: 2 + 2 = 4
- Translate y: 2 + 4 = 6
- Translate z: 2 + 6 = 4

$$\begin{pmatrix} 4 \\ 6 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

Translated point   **Translation Matrix**   **Original point**

General form

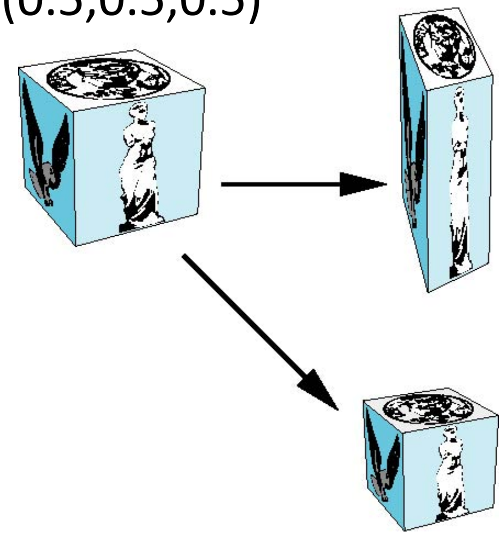
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Scaling Transform

- Expand or contract along each axis (fixed point of origin)

- **Example:** If we scale a point (2,4,6) by scaling factor (0.5,0.5,0.5)

Scaled point position = (1, 2, 3)



- Scaled x: 2 x 0.5 = 1
- Scaled y: 4 x 0.5 = 2
- Scaled z: 6 x 0.5 = 3

(2,4,6) x 0.5 = (1,2,3)

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 6 \\ 1 \end{pmatrix}$$

Scale Matrix for Scale(0.5, 0.5, 0.5)

General Form

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Why Matrices?

- Sequence of transform matrices can be pre-multiplied
- One final resulting matrix applied (efficient!)
- E.g. transform 1 x transform 2 ….

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Transformed Point

Transform Matrices can
Be pre-multiplied

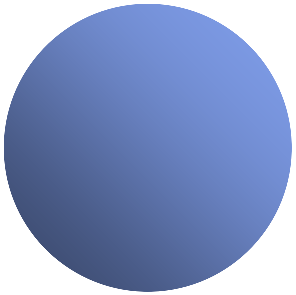
Original Point

- Computer graphics card has
  fast 4x4 matrix multiplier!!!

Why do we need Shading?

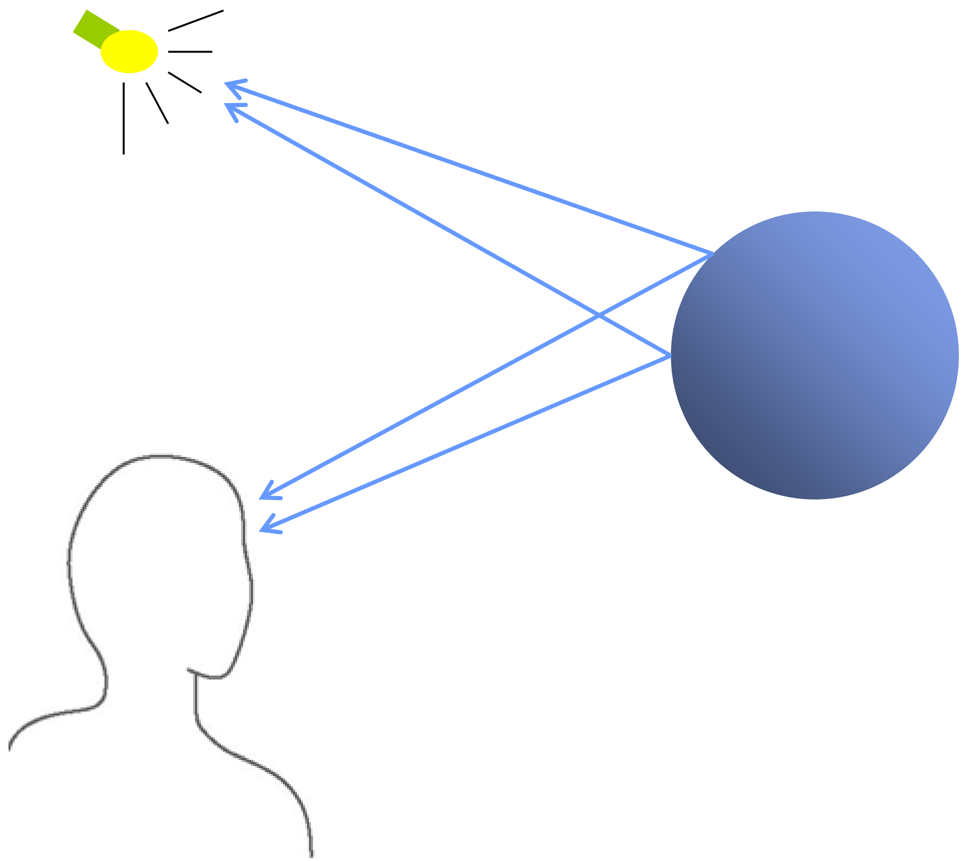
- Sphere without lighting & shading:

- Sphere with shading:
  - Has **visual cues** for humans (shape, light position, viewer position, surface orientation, material properties, etc)



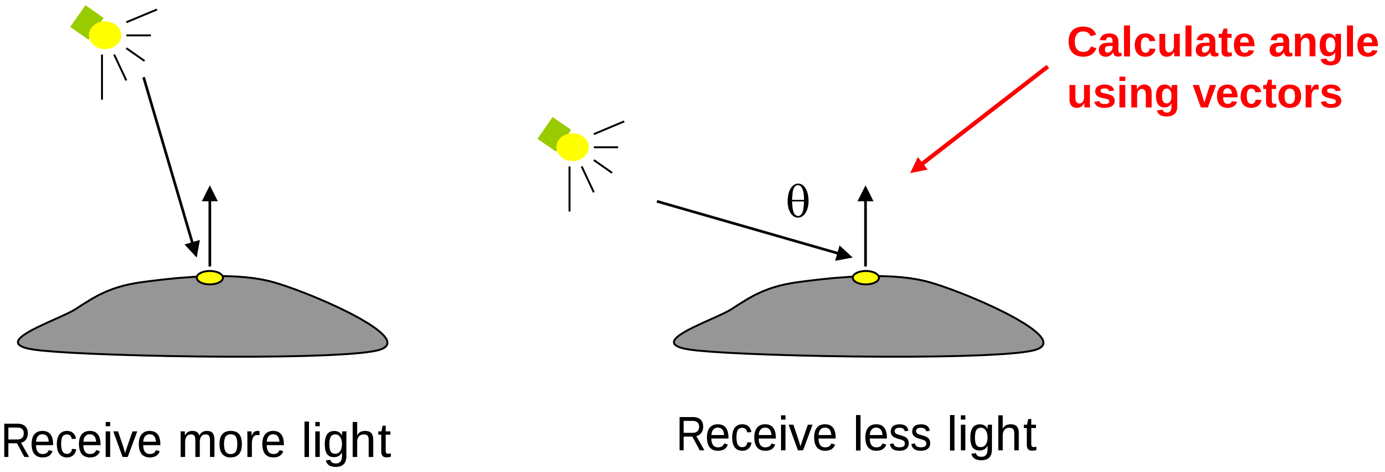
What Causes Shading?

- Shading caused by different angles with light, camera at different points



Calculating Shade

- Based on Lambert's Law: $D = I \times k_D \cos(\theta)$
  - Calculate shade based on angle θ



- Represent light direction, surface orientation as vectors
- Calculate θ? Angle between 2 vectors

Shading: Diffuse Light Example

Different parts of each object receives different amounts of light

References

- Angel and Shreiner, Interactive Computer Graphics (6th edition), Chapter 1
- Hill and Kelley, Computer Graphics using OpenGL (3rd edition), Chapter 1