Worcester Polytechnic Institute
Department of Mathematical Sciences May 2019

General Comprehensive Examination

Linear Algebra

Print Name: ________________________ Sign: ________________________

Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your **five best solutions**.

Problem 1: Consider $T : P_3 \to P_1$ defined by second differentiation, i.e., by $T(p) = p'' \in P_1$ for $p \in P_3$. Find the matrix representation of T with respect to the bases
$\{1 + x, 1 - x, x + x^2, x^2 - x^3\}$ for P_3 and $\{1, x\}$ for P_1.

Problem 2: Let A and B be $n \times n$ matrices. If $AB = 0$ show that $rank(A) + rank(B) \leq n$

Problem 3: Let $T : V \to V$ be a linear operator on the finite dimensional vector space V. Show that $rank(T) \geq rank(T^2)$. Show that if equality holds, then the intersection of $ker(T)$ and $image(T)$ is the zero vector.

Problem 4:

(a) Prove that the eigenvalues of a skew-symmetric matrix are either zero or (pure) imaginary.

(b) Let $\{\lambda_1, \ldots, \lambda_k\}$ be distinct eigenvalues of a linear operator $T : V \to V$. Prove that any corresponding set of eigenvectors $\{v_1, \ldots, v_k\}$ is linearly independent.

Problem 5: (Fredholm Alternative). Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:

(i) $Ax = b$

(ii) $A^T y = 0, \qquad y^T b \neq 0$

Problem 6: Let $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \\ 0 & 1 \end{pmatrix}$.

(a) Compute a singular value decomposition of A.

(b) Compute the vector $\hat{b}$ in the column space of A that is closest to $b = (1, 1, 1)^T$,