GCE: 502, Linear Algebra
May 2020

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Exercise 1:

Let A, B be two square complex matrices. Let $f_B(x) = det(xI - B)$ denote the characteristic polynomial of B. Prove that the matrix $f_B(A)$ is invertible if and only if A and B do not have any eigenvalue in common.

Exercise 2:

Let $T : V \to V$ be a linear transformation of a finite dimensional vector space.
(i). Prove the following chain of inclusions: $... \subset Range(T^3) \subset Range(T^2) \subset Range(T) \subset V$.
(ii). Prove that there exists a positive integer K such that $Range(T^k) = Range(T^{k-1})$ for all $k > K$.

Exercise 3:

Let V be a finite dimensional real inner product space and $T : V \to V$ be a self-adjoint isometry. Prove that there exists a subspace $U \subset V$ such that $T(x+y) = x - y$ for all $x \in U$ and y in the orthogonal complement of U.

Exercise 4:

Let $\top$ be the transpose operator acting on $n \times n$ matrices.
(i). Show that all eigenvalues of $\top$ belong to $\{\pm 1\}$.
(ii). Describe the eigenspaces of $\top$.
(iii). Show that τ is an orthogonal operator, and hence explain why every matrix has a unique decomposition as the sum of a symmetric and skew-symmetric matrix.

Exercise 5:

Suppose that $x^\top M x = \mathbf{0}$ for all vectors $x \in \mathbb{R}^n$.
(i). Prove that M is skew symmetric. Does every skew-symmetric matrix satisfy this condition?
(ii). If M is invertible, prove that n is odd.

Exercise 6:

Describe the dimensions n in which there exists an invertible matrix of multiplicative order 5, that is, a non-identity matrix M which satisfies $M^5 = I_n$. In which dimensions, is there

such a matrix for which 1 is not an eigenvalue?
Hint: Consider the matrix
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$