GCE: 503, Analysis and measure theory

May 2020

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Exercise 1:

Let X be a measure space $f_n : X \to \mathbb{R}$ a sequence of measurable functions, and $f : X \to \mathbb{R}$ a measurable function. By definition we say that f_n converges to f in measure if for all $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}),$$

converges to zero, as $n \to \infty$, where μ is the measure on X.

(i). Find a measure space X, $f_n : X \to \mathbb{R}$ a sequence of measurable functions, and $f : X \to \mathbb{R}$ a measurable function such that f_n converges to f almost everywhere but not in measure.

(ii). Find a measure space Y, $g_n : Y \to \mathbb{R}$ a sequence of measurable functions, and $g : Y \to \mathbb{R}$ a measurable function such that g_n converges to g in measure but not almost everywhere.

Exercise 2:

Let X be a metric space such that X is a finite set.

(i). Show that any convergent sequence in X is eventually constant.

(ii). Find (with proof) all subsets of X that are compact.

Exercise 3:

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or } x \in [0,1]\backslash\mathbb{Q}; \\ 1/q, & \text{if } x \in (0,1] \bigcap \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$

For instance, $f(0.75) = 1/4$ due to $0.75 = 3/4$ in lowest term; $f(1/\sqrt{2}) = 0$ due to $1/\sqrt{2} \notin \mathbb{Q}$.

(i). Is f a Lebesgue measurable function? Justify your answer.

(ii). Find $\int_0^1 f(x)dx$.

(iii). Prove that $f(x) \leq x$ for all $x \in [0, 1]$.

(iv). Find the set of points of discontinuity of f in $[0, 1]$.

Exercise 4:

Find (with proof) $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{\sin(x^n)}{x^n} dx.$

Exercise 5:

Let $f_n \in L^2([0, 1])$ and $f \in L^2([0, 1])$.

(i). Prove that, $\|f_n - f\|_2 \to 0$ implies that $\|f_n\|_2 \to \|f\|_2$.
(ii). Does $\|f_n\|_2 \to \|f\|_2$ imply $\|f_n - f\|_2 \to 0$? Justify your answer.
(iii). Suppose $\|f_n\|_2 \to \|f\|_2$ and $f_n \to f$ almost everywhere on $[0, 1]$. Show that $\|f_n - f\|_2 \to 0$.