Worcester Polytechnic Institute
Department of Mathematical Sciences

January 2021

General Comprehensive Examination

Linear Algebra

Print Name: ______________________ Sign: ______________________

No documents, no calculators allowed. Attempt all questions.

1. Let S_1, S_2 be k-dimensional subspaces of a finite dimensional vector space V over $\mathbb{R}$. Prove that there exists a subspace T such that $V = S_1 \oplus T = S_2 \oplus T$, i.e. that there exists a common complement of both spaces.

2. Let M be an $n \times n$ matrix with integer entries.

   (a) Prove or disprove: A prime p divides $\det(M)$ if and only if M, considered as a matrix with entries in the integers modulo p, is rank-deficient.

   (b) Let J_n be the $n \times n$ all-ones matrix, let I_n be the identity matrix. For integers a and b, prove that $\det(aJ_n + bI_n)$ is divisible by p if $a \equiv b \mod p$. Give an explicit example of an eigenvector for which the corresponding eigenvalue is divisible by p.

   (c) By row operations or otherwise, evaluate the determinant of $aJ_n + bI_n$ over $\mathbb{R}$. Hence give a necessary and sufficient condition for $\det(aJ_n + bI_n)$ to be divisible by a prime p.

3. Define the characteristic and minimum polynomials of an $n \times n$ matrix M. Describe the matrices for which the characteristic and minimum polynomials are equal.

4. Let $V = \mathrm{Sp}\{1, x, y, x^2, y^2, xy, x^3, x^2y\}$ be a subspace of the polynomial ring $\mathbb{R}[x, y]$. Construct a basis for V with respect to which the matrix of $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is in Jordan Canonical Form.

5. A matrix M with entries in $\mathbb{R}$ is *normal* if $MM^\top = M^\top M$.

   (a) Suppose that the eigenvectors of A form an orthonormal basis of $\mathbb{R}^n$. Prove that A is normal.

   (b) Suppose that B is upper-triangular and normal. Show that B is diagonal. (Hint: compare the norm of the i^{th} row with that of the i^{th} column.)

   (c) Do there exist diagonalisable matrices which are not normal? Give a proof or a counterexample.

6. Let

$$M = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 5 \\ -5/2 \\ 52 \end{bmatrix}.$$

   - Find the QR decomposition of M.
   - Find the least squares solution to $M\mathbf{x} = \mathbf{b}$