WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics - I
August, 2018

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. Let
$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\{-\frac{1}{2}(x^2 - 2\rho xy + y^2)\}, -\infty < x, y < \infty.$$
Find $f(y \mid x > 0)$. Deduce $f(y \mid x < 0)$.

2. Let $X, Y \overset{iid}{\sim}$ Uniform$(0, 1)$. Find $f(x, y)$ subject to $x < y$. Compute the cor(X, Y) subject to $X < Y$. What happens if instead $y < x$?

3. Let $X_1, \ldots, X_n$ be independent and identically distributed with finite variance, and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Assume that $f(x) = 2\Phi(x)\phi(x), -\infty < x < \infty$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are respectively the pdf and cdf of the standard normal random variable. Find T and the sequences μ_n and σ_n such that
$$\frac{\bar{X}_n - \mu_n}{\sigma_n} \xrightarrow{d} T.$$

4. Let $(X_1, \ldots, X_n)$ be a random sample from a Poisson distribution with mean θ . We are interested in estimating $g(\theta) = P(X_1 = 0) = e^{-\theta}$. Consider the following two estimators:
$$T_{n,1} = e^{-\bar{X}_n} \qquad \text{and} \qquad T_{n,2} = \frac{1}{n}\sum_{i=1}^n I(X_i = 0),$$
where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, and I is an indicator function.
   (a) Find the asymptotic distribution of $T_{n,1}$.
   (b) Find the asymptotic distribution of $T_{n,2}$.
   (c) Which estimator is more efficient in estimating $g(\theta)$ when a large sample size is available? Show your argument.

5. Let $P_1, ..., P_n$ be n i.i.d. Uniform(0,1) random variables, and $P_{(1)} \le P_{(2)} \le ... \le P_{(n)}$ are their order statistics. For a given $k \in \{1, 2, ..., n\}$, define
$$W_k = \prod_{i=1}^k P_{(i)}.$$
Find the CDF of W_k.

(a) Define $U_i = \frac{P_{(i)}}{P_{(i+1)}}$, $i = 1, ..., k-1$. Show that $P_{(i)} \sim \text{beta}(i, n-i+1)$ and $U_i^i \sim \text{Uniform}(0,1)$.

(b) Show that the random variables $U_1, U_2^2, ..., U_{k-1}^{k-1}$, and $P_{(k)}^k$ are all independent to each other.

(c) Get the formula for $P(W_k \leq w)$, where $w \in (0,1)$ is a constant.

6. Let $T = X + Y$, where X and Y are independent random variables.

   (a) Suppose $X \sim \text{Gamma}(\alpha, \beta)$ and $T \sim \text{Gamma}(\nu, \beta)$ with $\nu > \alpha$. Find the distribution of Y. Explain.

   (b) Suppose $T \sim \text{Normal}(\mu, \sigma^2)$ and $X \sim \text{Normal}(\theta, \delta^2)$. What condition is necessary for Y to have a normal distribution? Explain.

   (c) Give an example that shows T is independent of $X - Y$.