WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 541 Probability and Mathematical Statistics II

January, 2021

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. Let the pdf of a logistic location distribution be

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}\ ,\ -\infty < x < \infty, -\infty < \theta < \infty.$$

   (a) (10 points) Does this distribution family have a monotone likelihood ratio (MLR)?

   (b) (10 points) Find the uniformly most powerful (UMP) size α test for the hypotheses $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$ based on an observation.

2. (20 points) Let $X_1, ..., X_n$ be iid Poisson(λ), and λ has a prior distribution gamma(α, β). Assume that α and β are known constants. What is the Bayes estimator of λ?

3. (20 points) Let X_i be i.i.d. Bernoulli(p), $i = 1, 2, ...n$. We know $\bar{X}_n(1-\bar{X}_n)$ is the MLE of the variance of X_i. How is $\bar{X}_n(1-\bar{X}_n)$ distributed asymptotically as $n \to \infty$?

4. (20 points) Suppose $X_1, X_2, \cdots, X_n$ are *iid Bernoulli*(p) where $n \geq 2$ and $0 < p < 1$ is the unknown parameter.

   a) Derive the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(p)$, where $\tau(p) = e^2(p(1-p))$.

   b) Find the Cramer-Rao lower bound (CRLB) for estimating $\tau(p) = e^2(p(1-p))$.

5. (20 points) Let $X_1, \cdots, X_n$ be iid with common density

$$f(x|\theta, \lambda) = \lambda e^{-\lambda(x-\theta)}, \quad x \geq \theta,$$

   where $\theta > 0$ and $\lambda > 0$ are two parameters.

   1. Find a maximum likelihood estimator for (θ, λ) based on the sample $(X_1, \ldots, X_n)$.
   2. Let $X_{(1)} = \min_{i=1,\ldots,n} X_i$. Assume that $\lambda = 1$ and θ is unknown. Consider a class of confidence intervals for θ that have the form $(X_{(1)} - c_1, X_{(1)} - c_2)$ where $0 \leq c_2 < c_1$ are two constants. For fixed confidence level $1 - \alpha$, find c_1 and c_2 such that the confidence interval length $c_1 - c_2$ is minimized.

6. (20 points) Let X, Y be two binary variables with

$$P(X = r, Y = s) = \pi_{rs}, \quad r, s \in \{0, 1\}.$$

a. Find $\rho = \mathrm{Corr}(X, Y)$.
b. For a sample of size n, we observe counts N_{rs}, where $r, s \in \{0, 1\}$. Find the maximum likelihood estimator of ρ.