WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics - II
August, 2018

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. Let $X_1, \ldots, X_n \mid \beta \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$, where α is assumed known. Let $R = S/\bar{X}$, where $\bar{X}$ and S are respectively the sample mean and sample standard deviation. What parameter does R estimate? Is R an unbiased estimator? Argue that R is stochastically independent of $\bar{X}$. Find $\mathrm{E}(R^2)$ and give an approximation to $\mathrm{E}(R)$. [For $X \sim \text{Gamma}(a, b), \mathrm{E}(X) = a/b, \mathrm{Var}(X) = a/b^2$.]

2. Let $X_1, \ldots, X_n \mid \theta$ be independent and identically distributed with
$$f(x \mid \theta) = \frac{2x}{\theta^2}, 0 < x < \theta.$$
Find the MLE, $\hat{\theta}$, of θ and its asymptotic distribution (as n goes to infinity). What optimality property does $\hat{\theta}$ have?

3. A forest has N (unknown) monkeys. A random sample of n monkeys is selected from the forest, tagged and released back into the forest. After a few days, a random sample of m monkeys is selected and Y monkeys are found with the tags. Find an estimator of N. Approximate the mean and variance of your estimator. In this experiment, with $n = 25$ and $m = 16$, Y was observed to be 8. Find an approximate 95% confidence interval for N.

4. Let $Y_1, \ldots, Y_n$ be iid from a one parameter exponential family with pdf or pmf $f(y|\theta)$ with complete sufficient statistic $T(Y) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$ and X has a known distribution with known $E(X)$ and known variance $V(X)$. Let $W_n = cT(Y)$ be an estimator of θ where c is a constant.
   (a) Find the mean square error (MSE) of W_n as a function of c (and of n, $E(X)$ and $V(X)$).
   (b) Find the value of c that minimizes the MSE. Prove that your value is the minimizer.
   (c) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

5. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ.
   (a) Show that $T = \sum_{i=1}^n X_i$ is complete sufficient statistic for θ. .
   (b) For $a > 0$, find the uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta) = e^{a\theta}$.

(c) Prove the identity:

$$E\left[2^{X_1}|T\right] = \left(1 + \frac{1}{n}\right)^T.$$

6. Let $X_1, \ldots, X_n \overset{iid}{\sim}$ Gamma$(1, \theta)$ and independently $Y_1, \ldots, Y_m \overset{iid}{\sim}$ Gamma$(1, \mu)$. Consider testing $H_o : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

   (a) Find the likelihood ratio test (LRT).

   (b) Let $T = \frac{n\bar{X}}{n\bar{X}+m\bar{Y}}$. Show that the LRT can be based on T.

   (c) Find the distribution of T when H_o is true.