The Second Electrolyte Wedge Problem

Kimberly Ware

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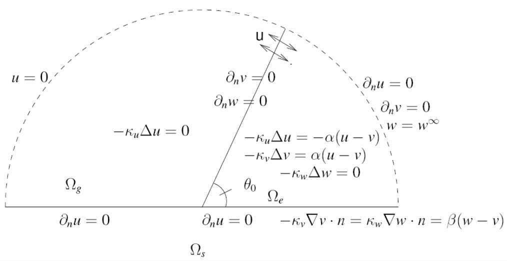
Abstract –

The Second Electrolyte Wedge problem studies diffusion-reaction-conduction processes associated with current production in a porous electrode. Two rate-determining reaction steps occur in this formulation – one in the electrolyte wedge and one at the electrolyte-solid interface. Existence and uniqueness of solutions to this problem are proven, and thus current density is proven to be finite. Numerical and asymptotic analysis are completed and expressions for the current density and total current produced by the electrolyte are given.

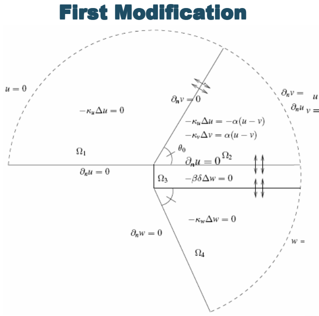
1. Introduction

The Second Electrolyte Wedge Problem

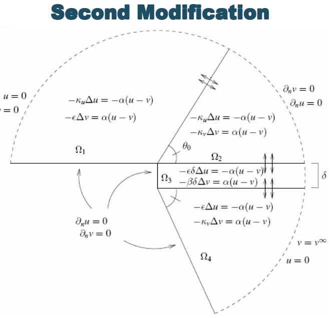


2. Equivalent Problems

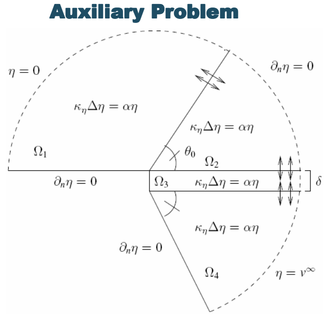
First Modification



Second Modification



Auxiliary Problem



Theorem 1

There exists a unique solution to the *Auxiliary Problem.*

Theorem 2

There exists a unique solution to the *second modification.* Moreover, the current densities (potential gradients) are bounded.

3. Theorem 1: Existence Proof

Overview of the Proof

$E[\eta] := \int_\Omega \kappa_\eta |\nabla \eta|^2 + \alpha \eta^2 \, dx$ **Minimizer of energy gives existence of solution**

Proof depends on:

- **Coercivity**
- **Convexity**
- **Weak lower semicontinuity**

4. Theorem 1: Uniqueness Proof

- Assume two solutions (η_1 and η_2)
- Let $\omega = \eta_1 - \eta_2$
- $\kappa_\eta \triangle \omega = \alpha \omega$ in Ω and $\frac{\partial \omega}{\partial n} = 0$ or $\omega = 0$ everywhere on $\partial\Omega$
- Using Green's first identity:

$$\int_{\partial\Omega} \omega \frac{\partial \omega}{\partial n} \, dS = \int_\Omega |\nabla \omega|^2 \, dx + \int_\Omega \alpha \omega^2 \, dx$$

- Using the boundary conditions:

$$\int_\Omega |\nabla \omega|^2 \, dx + \int_\Omega \alpha \omega^2 \, dx = 0$$

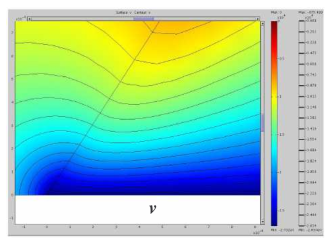
- Combining gives:

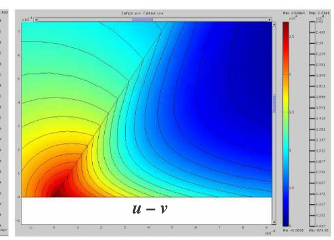
$$\int_\Omega |\nabla \omega|^2 + \alpha \omega^2 \, dx = 0$$

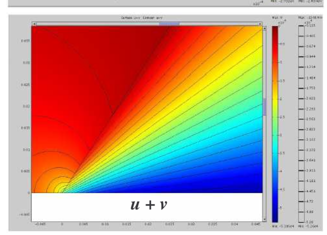
- Integrand nonnegative, $\alpha \geq 0$, and ω continuous implies $\omega = 0$

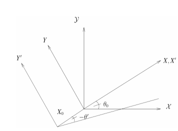
Proofs of existence and uniqueness of the Auxiliary Problem make the Second Modification a Poisson problem in u and v which will also have a unique solution.

5. Numerical & Asymptotic Results









A change of coordinate systems is introduced to help with asymptotic calculations.

Asymptotic Results

- $X_0 = \frac{2\cos\theta_0}{\theta_0}$
- $u_0 = \frac{w^\infty}{\theta_0} \cot^{-1}\left(\frac{\epsilon_2 X_0}{\beta r} \csc(\theta_0 - \theta) + \cot(\theta_0 - \theta)\right)$
- $w_0 = w^\infty$
- $v_0 = \frac{w^\infty}{\theta_0} \cot^{-1}\left(\frac{\epsilon_2 X_0}{\beta r} \csc(\theta_0 - \theta) + \cot(\theta_0 - \theta)\right)$

Current Density and Total Current

$$i_F = \beta(w - v)/F = \frac{\beta w^\infty}{F\theta_0}\left[\theta_0 - \tan^{-1}\left(\frac{\tan(\theta_0)}{1 + \epsilon/(\theta_0 \beta r)}\right)\right] + O(\epsilon, \theta_0^2)$$

$$\int_0^{r^\infty} i_F dr = \frac{\varepsilon w^\infty}{F\theta_0}\left[R^\infty\left(\theta_0 - \tan^{-1}\left(\frac{\tan\theta_0}{1 + 1/(\theta_0 R^\infty)}\right)\right) + \frac{X_0 \sin\theta_0}{2} \ln\left(\left(\frac{R^\infty}{X_0}\right)^2 + 2R^\infty\theta_0 + 1\right) - X_0\cos\theta_0\left(\theta_0 - \tan^{-1}\left(\frac{\tan\theta_0}{1 + R^\infty/(X_0\cos\theta_0)}\right)\right)\right] + O(\epsilon^2, \theta_0^2),$$

where $R^\infty := r^\infty \beta/\varepsilon$

6. Conclusions

- **Total current produced by wedge has been approximated**
- **This could be used to approximate total current produced by electrode**

Advisor: Prof. Joseph D. Fehribach