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Term C03

PH 3402: Notes 4

The theory of angular momentum is dealt with in Griffiths Ch.4. I just emphasize some important points. It was shown in lecture how all the properties of angular momentum in quantum mechanics follow from the basic commutation rule $\left[L_x, L_y\right] = i\hbar L_z$ plus its cyclic variants. The raising and lowering operators are defined as $L_\pm = L_x \pm iL_y$. The most important relations involving the angular momentum operators and eigenstates are:

$$L^2\left|l,m\right\rangle = \hbar^2 l\left(l+1\right)\left|l,m\right\rangle, \quad L_z\left|l,m\right\rangle = \hbar m\left|l,m\right\rangle, \quad L_\pm\left|l,m\right\rangle = \hbar\sqrt{l(l+1)-m(m\pm 1)}\ \left|l,m\pm 1\right\rangle \quad (1)$$

The eigenstates form an orthonormal and complete set:

$$\left\langle l,m\middle|l',m'\right\rangle = \delta_{ll'}\delta_{mm'} \qquad (2a)$$

$$\sum_{m=-l}^{l}\left|l,m\right\rangle\left\langle l,m\right| = I \qquad (2b)$$

where the second relation says that the $2l+1$ eigenstates for a particular l form a complete set for that particular l. Thus I on the right side of (2b) is the identity operator in this subspace of the larger space spanned by all the eigenstates.

The above relations allow one to obtain matrix representations of the angular momentum operators as square l x l matrices in the basis afforded by the angular momentum eigenstates for a fixed value of l.

I now discuss three different aspects of the angular momentum operators/eigenstates.

<u>Uncertainty relations</u>

First note that the expectation values of the operators in the standard eigenstates are

$$\left\langle l,m\middle|L_z\middle|l,m\right\rangle = \hbar m, \quad \left\langle l,m\middle|L_x\middle|l,m\right\rangle = \left\langle l,m\middle|L_y\middle|l,m\right\rangle = 0 \qquad (3)$$

The last two relations follow if one expresses L_x and L_y in terms of the raising and lowering operators and uses (1) and (2a). We also have the result

$$\left\langle l,m\middle|L_x^2\middle|l,m\right\rangle = \left\langle l,m\middle|L_y^2\middle|l,m\right\rangle = \frac{\hbar^2}{2}\left[l(l+1)-m^2\right] \qquad (4)$$

This can be proved by expressing L_x, L_y in terms of $L_\pm$ and using (1) or, more simply, by noting that $L_x^2 + L_y^2 = L^2 - L_z^2$ and that $\left\langle l,m\middle|L_x^2\middle|l,m\right\rangle = \left\langle l,m\middle|L_y^2\middle|l,m\right\rangle$ from symmetry.

Recall the generalized uncertainty principle for two non-commuting observables A and B, according to which the product of their variances obeys the inequality

$$\Delta A \bullet \Delta B \geq \frac{1}{2i}\left|\left\langle [A,B] \right\rangle\right| \qquad (5)$$

where the variances and the expectation value are all to be calculated in the same quantum state $|\psi\rangle$. Let us apply this relation to the case $A = L_z, B = L_x$ and $|\psi\rangle = |l,m\rangle$. Then we have

$$\begin{aligned}
\Delta A &= \Delta L_z = \sqrt{\langle l,m|L_z^2|l,m\rangle - \langle l,m|L_z|l,m\rangle^2} = 0 \\
\Delta B &= \Delta L_x = \sqrt{\langle l,m|L_x^2|l,m\rangle - \langle l,m|L_x|l,m\rangle^2} = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1)-m^2} \qquad (6) \\
\langle l,m|[L_z,L_x]|l,m\rangle &= i\hbar\langle l,m|L_y|l,m\rangle = 0
\end{aligned}$$

from which we see that both the left and right sides of (5) are zero and that the uncertainty relation is satisfied as an equality. A more interesting case occurs if we take $A = L_x, B = L_y$ and $|\psi\rangle = |l,m\rangle$. Then one finds that

$$\begin{aligned}
&\Delta L_x = \Delta L_y = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1)-m^2} \\
\text{and} \quad &\frac{1}{2i}\langle l,m|[L_x,L_y]|l,m\rangle = \frac{\hbar}{2}\langle l,m|L_z|l,m\rangle = \frac{\hbar^2}{2}m
\end{aligned} \qquad (7)$$

and (5) reduces to the inequality $l(l+1) \geq m^2 + |m|$. Since $-l \leq m \leq l$, this inequality is always satisfied and the uncertainty principle is verified to be true. One also sees that the only states that are minimum uncertainty states are the two extremal states $|l,\pm l\rangle$.

The above analysis justifies the "vector model" of angular momentum sometimes depicted in text books. We have seen that the eigenstate $|l,m\rangle$ has sharp values of L^2 and L_z associated with it but that L_x and L_y are both uncertain and have average values of 0. This state of affairs can be captured geometrically by representing the angular momentum operator by a vector of fixed length that sweeps out a cone about the z-axis. The constancy of the length of the vector and its projection on the axis of the cone model the sharpness of L^2 and L_z, while the rotation about the axis of the cone models the washing out of a definite value of L_x or L_y and also the averaging of these values to zero (over a cycle of rotation).

<u>Central potentials</u>

A central potential $V(r)$ depends only on the radial distance from the center of force (generally taken at the origin) and has complete spherical symmetry about this center. We want to show that the Hamiltonian for a central potential, $H = p^2/2m + V(r)$, commutes with both L^2 and L_z and that one can therefore construct simultaneous eigenstates of all these observables (which is what is usually done). To show that L_z commutes with H, we show that it commutes individually with the operators p^2 and $V(r)$; then we do likewise for L^2.

$$
\begin{aligned}
\left[L_z, p^2\right] &= \left[L_z, p_x{}^2\right] + \left[L_z, p_y{}^2\right] + \left[L_z, p_z{}^2\right] \\
&= \left[L_z, p_x\right] p_x + p_x \left[L_z, p_x\right] + \left[L_z, p_y\right] p_y + p_y \left[L_z, p_y\right] + \left[L_z, p_z\right] p_z + p_z \left[L_z, p_z\right] \\
&= \left(i\hbar p_y\right) p_x + p_x \left(i\hbar p_y\right) + \left(-i\hbar p_x\right) p_y + p_y \left(-i\hbar p_x\right) + 0 + 0 = 0
\end{aligned}
$$

The commutators in the second line can be evaluated by writing out the components of the angular momentum operator in terms of the position and momentum operators and then invoking the fundamental commutation rules.

Next we evaluate the commutator of L_z and V by expressing L_z as a differential operator in the coordinate representation:

$$
\begin{aligned}
\left[L_z, V\right] f &= L_z V f - V L_z f \\
&= \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) V f - \frac{\hbar}{i} V \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) f \\
&= \frac{\hbar}{i}\left(x\frac{\partial (Vf)}{\partial y} - y\frac{\partial (Vf)}{\partial x}\right) - \frac{\hbar}{i} V \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) \\
&= \frac{\hbar}{i}\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right) f = \frac{\hbar}{i}\left(\vec{r}\times\nabla V\right)_z f = 0
\end{aligned}
$$

where we used the fact that, for a central potential, ∇V is along $\vec{r}$ and hence that $\vec{r}\times\nabla V = 0$. Since the above holds for an arbitrary function f, it follows that $\left[L_z, V\right] = 0$.

By expressing L^2 as $L_x^2 + L_y^2 + L_z^2$ and making use of results similar to those derived above, it can be shown without much difficulty that L^2 also commutes with both p^2 and V and hence with H. This proves everything we wanted to.

Constructing the spherical harmonics the easy way

Here is an easy way to construct the spherical harmonics $Y_l^m\left(\theta,\phi\right) = \left\langle \theta,\phi \middle| l,m \right\rangle$ without having to solve a complicated differential equation. Begin from the observation that the highest spherical harmonic Y_l^l in a multiplet is annihilated by the raising operator L_+ and is also an eigenstate of L_z with eigenvalue $l\hbar$:

$$
L_+ Y_l^l = 0 \quad \text{and} \quad L_z Y_l^l = \hbar l Y_l^l \qquad (8)
$$

Using the fact that $L_z = -i\hbar \dfrac{\partial}{\partial \phi}$, the latter equation in (8) becomes the differential equation

$$-i\hbar \frac{\partial}{\partial \phi} Y_l^l(\theta,\phi) = \hbar l Y_l^l(\theta,\phi) \qquad (9)$$

which is easily solved by separation of variables to yield $Y_l^l(\theta,\phi) = y(\theta)e^{il\phi}$. On using the differential representation of L_+ (see Griffiths Eqn.[4.130]), the first equation in (8) can be recast as the differential equation

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial \theta} + i\cot\theta \frac{\partial}{\partial \phi}\right) Y_l^l(\theta,\phi) = 0 \quad \text{or} \quad \frac{dy}{d\theta} - l\cot\theta\, y = 0$$

whose solution is easily seen to be $y = \sin^l \theta$. Thus we have that $Y_l^l(\theta,\phi) = N_l \sin^l \theta e^{il\phi}$, where N_l is a normalization constant (that depends on l). You can easily work out the constant for small l by requiring that the spherical harmonic be normalized over the unit sphere. You can then work out all the other harmonics in the same multiplet by repeatedly applying the lowering operator to the topmost harmonic. The basic relation you need is

$$L_- Y_l^m = \hbar\sqrt{l(l+1) - m(m-1)}\, Y_l^{m-1}$$

which, upon use of the differential representation of L_- (see Griffiths Eqn.[4.130]), becomes

$$-e^{-i\phi}\left(\frac{\partial}{\partial \theta} - i\cot\theta \frac{\partial}{\partial \phi}\right) Y_l^m(\theta,\phi) = \sqrt{l(l+1) - m(m-1)}\, Y_l^{m-1}(\theta,\phi)$$

With $m = l$ the above relation will allow you to calculate Y_l^{l-1} from Y_l^l and then you can iterate the procedure to go downwards.

Exercise: Use this procedure to calculate all the spherical harmonics for $l = 1, 2$ by first fixing the highest harmonic and then using the lowering operator to proceed downwards. Check your answers against those given in Table 4.2 of Griffiths. Note: you could also have first calculated Y_l^{-l} and then used the raising operator to proceed upwards.

Differential equations for the spherical harmonics

I show how one makes the transistion from the eigenvalue equation (8) to the differential equation (9), from which you can also appreciate the analogous transition for the equation for L^2. First note the result $\langle x|\hat{p}_x|\psi\rangle = -i\hbar \dfrac{\partial}{\partial x}\psi(x)$, which can be proved as follows:

$$\langle x|\hat{p}_x|\psi\rangle = \left\langle x\left|\hat{p}_x\left(\int dp_x\,|\,p_x\rangle\langle p_x|\right)\right|\psi\right\rangle = \int dp_x\,p_x\,\langle x|\,p_x\rangle\langle p_x|\psi\rangle$$
$$= -i\hbar\frac{\partial}{\partial x}\int dp_x\,\langle x|\,p_x\rangle\langle p_x|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x)$$

The explanation of the above manipulations is as follows: in the second step we inserted the completeness relation for the x-component momentum eigenstates, in the third step we replaced the operator $\hat{p}_x$ by its eigenvalue p_x, in the fourth step we used the fact that

$$p_x\,\langle x|\,p_x\rangle = p_x\exp\left(\frac{i}{\hbar}p_x x\right) = -i\hbar\frac{\partial}{\partial x}\exp\left(\frac{i}{\hbar}p_x x\right) = -i\hbar\frac{\partial}{\partial x}\langle x|\,p_x\rangle$$

and in the fifth step we "folded back" the completeness relation to the identity operator.

With this in hand, we can show that

$$\langle x,y,z|L_z|\psi\rangle = \left\langle x,y,z\left|\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right|\psi\right\rangle = x\left\langle x,y,z\left|\hat{p}_y\right|\psi\right\rangle - y\left\langle x,y,z\left|\hat{p}_x\right|\psi\right\rangle$$
$$= \left[x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\psi\left(x,y,z\right) = \left[-i\hbar\frac{\partial}{\partial\phi}\right]\psi(r,\theta,\phi) \qquad (10)$$

where we switched to spherical coordinates in the last step, with the quantity in brackets being the differential operator form of L_z in the coordinate representation. Now take the eigenvalue equation $L_z\left|l,m\right\rangle = \hbar m\left|l,m\right\rangle$ and project it onto the coordinate representation to get $\langle r,\theta,\phi|L_z\left|l,m\right\rangle = \hbar m\langle r,\theta,\phi|l,m\rangle$. Define $\langle r,\theta,\phi|l,m\rangle = R(r)Y_l^m\left(\theta,\phi\right)$ i.e., a product of radial and angular functions. On using (10) to reexpress the left side and canceling off the radial function $R(r)$, we obtain the general form of the differential equation (9) for an arbitrary spherical harmonic Y_l^m.