Bifurcations in the Rayleigh-Benard Problem

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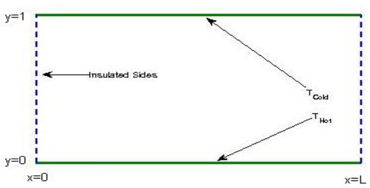
Abstract –

Thermal convection is the flow of fluid induced by a temperature difference, or gradient. Thermal convection has been studied for the past two hundred years, but analytical solutions are lacking for many important situations. Rayleigh-Benard convection is a particular type of thermal convection problem. This type of convection occurs due to the competing forces of gravity and buoyancy, which create an instability in the flow that induces fluid motion. Some bifurcation analysis has been performed on the system in the past, but the results were based on linearized equations. The goal of this project was to analyze the system's bifurcation behavior computationally, both to verify analytical and experimental results and to develop new conclusions regarding system behavior as the parameters are varied.

1. Introduction

A picture illustrating the physical problem is shown below. The rectangular domain is insulated on the sides, while a vertical temperature gradient is maintained by specifying hot and cold temperature values, respectively, at the bottom and top of the domain.

The two governing parameters in the system are the Rayleigh number, which is a function of the physical parameters of the system (density, temperature values, and dimensions), and the aspect ratio, which is the ratio of width to height.



2. The Model

- **The problem is modeled using the Navier-Stokes equations with the Boussinesq approximation.**
- **Analytical solutions are not known; numerical solution techniques are necessary.**

$$\frac{1}{P_r}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\omega = R_a\frac{\partial T}{\partial x}+\Delta\omega$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = -\frac{\partial\psi}{\partial x}+\Delta T$$

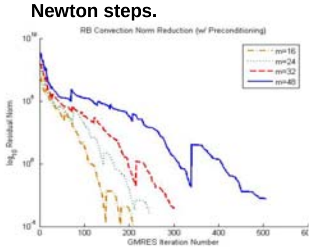
$$\Delta\psi = -\omega$$

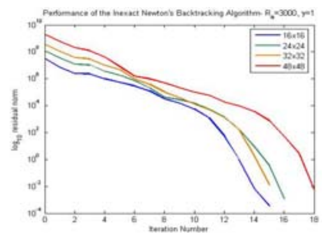
3. Assumptions

1. **The fluid is incompressible.**
2. **Thermal conductivity, viscosity, and the pressure coefficient are all constant.**
3. **Variation of density is only significant in the buoyancy term.**
4. **Viscous dissipation is negligible.**
5. **Density in the buoyancy term is linear with respect to temperature.**

4. Numerical Methods

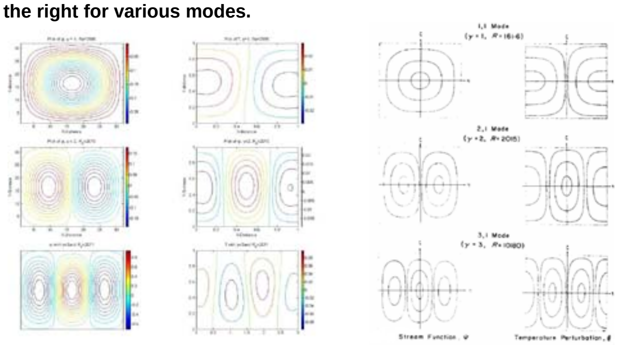
1. **The equations were discretized using finite-differences.**
2. **The nonlinear system was solved using an inexact Newton backtracking method.**
3. **The iterative linear solver GMRES was used to compute inexact Newton steps.**



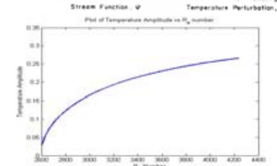


A Sample of Computational Results

Results from the model and [1] are presented below. The plots show the stream function on the left and the temperature distribution on the right for various modes.



Solution amplitude as a function of Ra (shows bifurcation point) ⇨



5. Conclusion

- **The computational results agreed with both experimental and analytical work.**
- **The linear analytical model frequently used does not truly capture system behavior, as seen in the discrepancies between the computational and analytical results above.**
- **The computational results predict a bifurcation of the first modal solution around R_a = 2600, which agrees with experimental and analytical results.**

Reference

1. **U.H. Kurzweg, Convective instability of a hydromagnetic fluid within a rectangular cavity, *Int. J. of Heat Mass Transfer*, vol. 8, p. 35, 1965.**

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