

**Proofs of the Kochen-Specker Theorem in 3 Dimensions**

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## **Abstract**

This project examines two sets of rays in three dimensions that prove the Bell-Kochen-Specker theorem, the 33 rays of Peres and the 33 rays of Penrose. It is shown that these rays possess identical orthogonality tables and thus lead to identical proofs of the theorem. It is also shown that these sets of rays are particular members of a three parameter family of 33 rays that prove the theorem.

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## **Chapter 1: Introduction**

Quantum mechanics, formulated in the early 20<sup>th</sup> century, is an exceptionally successful theory of nature. It accounts for phenomena in the microscopic world of atoms and molecules, where the laws of classical physics fail. Quantum mechanics has been confirmed by numerous experiments. However, the view of reality that it advocates differs radically from that of classical physics.

Quantum mechanics describes a particle by means of a wave function that is a complex function of space and time. However, even if one knows the wave function of a particle exactly, one does not have a definite knowledge of all its properties. In general, one can predict only the probabilities of the different values of a physical property, such as position or momentum. Quantum mechanics even goes so far as to say that a particle generally does not have a definite value of a physical property, such as position or momentum, unless a measurement on the particle is made, and that the observed value only comes into being as a result of the measurement. Some prominent physicists, such as Einstein, felt that quantum mechanics was a fundamentally unsatisfactory theory and would eventually be replaced by a superior “hidden variables theory” that would ascribe definite physical properties to systems even in the absence of any measurement.

However the Irish physicist, J. S. Bell [1], proved a mathematical theorem, now known as Bell's theorem, showing that hidden variable theories are impossible. He did this by proposing a class of experiments for which quantum mechanics and the hidden variable theories make very different predictions. Such experiments were carried out in the 1980s and 90s, and showed that the hidden variable theories were wrong and quantum mechanics was right.

Closely related to Bell's Theorem is the Bell-Kochen-Specker (BKS) theorem, proved by Bell [2] and by Kochen and Specker [3] in the late 1960s. This theorem rules out a more restricted class of hidden variable theories known as non-contextual hidden variable theories. It is the BKS theorem that

is the subject of this project. Specifically, this project will examine two proofs of the theorem for a spin-1 particle, one given by Peres [4] and the other by Penrose [5], and reveal the close connection between them.

The rest of the introduction will review some basic facts about spin 1 particles. Then a brief explanation will be given of the BKS theorem, following which the goals of the project, and the principal achievements, will be laid out.

### Three State Systems

An arbitrary state of a spin-1 system can be represented by a vector  $\psi = (c_1, c_2, c_3)$  with three complex components. The two states  $\psi_1 = (a_1, a_2, a_3)$  and  $\psi_2 = (b_1, b_2, b_3)$  are said to be orthogonal if  $a_1^* b_1 + a_2^* b_2 + a_3^* b_3 = 0$ . Three states  $\psi_1, \psi_2$  and  $\psi_3$  are said to form a basis if each pair of these states in them are orthogonal. There are an infinite number of bases of a spin-1 system, and the proof of the BKS theorem involves looking at a large number of states that give rise to partially overlapping bases, as explained below.

### The BKS Theorem

The quantum theory represents any set of compatible measurements (i.e. measurements that can be carried out using the same experimental arrangement) by a set of commuting hermitian operators. The simultaneous eigenstates of these commuting operators give rise to an orthonormal basis, with each state of the basis corresponding to one of the possible outcomes of the measurement. The orthodox (Copenhagen) interpretation of quantum mechanics holds that the observed state does not exist before the measurement, but comes into being only as a result of the measurement. However, the “hidden variables” theories hold that the state existed even before the measurement. Is there any way of deciding which viewpoint is correct?

If one takes hidden variables theories seriously, then, for any set of compatible measurements, only one of the corresponding basis states exists (and reveals itself upon measurement) while the other two do not. Let us assign the color green to the state that exists and red to the two states that do not. Some incomplete measurements allow us to distinguish only between two orthogonal states and not three. We will refer to this pair of states as a “dyad”. Since at most one of the states in a dyad can be revealed by a measurement (but neither need be), a dyad can have at most one green state in it.

One can make a large number of compatible measurements on a spin-1 system, some of which could be incomplete. Correspondingly, one will have a large number of bases and “dyads”. A hidden variables theory would assign a definite color to each state, either green or red, depending on whether it exists before the measurement or not. A non-contextual hidden variables theory would assign the same color to a state no matter which basis or dyad it is considered a part of. Consider a number of states that form several partially overlapping bases and dyads. A proof of the BKS theorem can be given by showing that it is impossible to color the states red or green in such a way that each basis has exactly one green ray in it while no dyad has two greens in it. The impossibility of this coloring would rule out the viability of any (non-contextual) hidden variables theory.

In 1966 Kochen and Specker [3] came up with a set of 117 states of a spin-1 system that could not be colored according to the above rules and so proved the BKS theorem. Kochen and Conway [6] later came up with a set of just 31 states that proved the theorem. This project looks at two three dimensional sets of 33 rays based on the geometry of a cube, one due to Peres [4] and the other due to Penrose [5], that prove the theorem. The Peres rays are real while the Penrose rays are complex. This report shows that while the two sets appear different and in fact are not unitarily equivalent to each other, they nevertheless possess identical orthogonality tables and so give rise to isomorphic proofs of the BKS theorem.

Chapter 2 will discuss some technical matters concerned with the description of spin-1 quantum systems. Chapter 3 will discuss the Peres and Penrose rays and their orthogonalities, and then demonstrate their isomorphism (as far as their orthogonality tables are concerned). Following that, a combined non-colorability proof of the BKS theorem is given for both of them, as well as a combined proof of the criticality of these sets. In addition, the common orthogonality table of both these sets is taken as a starting point and used to determine the general structure of any set of rays that is consistent with them. Chapter 4 summarizes the results of this project and also points out some remaining issues to be examined.

## Chapter 2: Descriptions of Spin-1 systems

There are several ways to represent the state of a spin-1 (or three state) quantum system. This chapter discusses two such representations that will be used in this project. These two will be referred to as the standard representation and the Majorana representation. For the work in the next chapter, it is important to know how to switch back and forth between these two descriptions.

A convenient set of basis states for a spin-1 system are the eigenstates of the spin operator along a particular direction, usually taken to be the z-axis. Labeling the states by the value of the spin projection in units of  $\hbar$ , the basis states are  $|+1\rangle$ ,  $|0\rangle$  and  $|-1\rangle$ , and the most general state of the system can be written as

$$|\psi\rangle = c_1|+1\rangle + c_2|0\rangle + c_3|-1\rangle, \quad (2.1)$$

where  $c_1$ ,  $c_2$  and  $c_3$  are complex numbers with the property that  $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$ . The state can also be written more briefly as  $(c_1, c_2, c_3)$ . Since neither the normalization nor the overall phase of a state are important for this project, we will often multiply the state  $(c_1, c_2, c_3)$  by a common factor to make each the components as simple as possible. As noted in chapter 1, two states  $\psi_1 = (a_1, a_2, a_3)$  and  $\psi_2 = (b_1, b_2, b_3)$  are orthogonal if  $a_1^*b_1 + a_2^*b_2 + a_3^*b_3 = 0$ , and any three mutually orthogonal states form a basis.

A second description of a spin-1 system can be given using a formalism developed by the Italian physicist E. Majorana [7]. In this approach, any state of the system is represented by a pair of points on the unit sphere. If  $\hat{a}_1$  and  $\hat{a}_2$  are the unit vectors to these points, the state can be specified in terms of them as follows. First, one maps these points on to the complex plane by performing a stereographic projection through the south pole of the sphere on to the plane passing through its equator. This yields the two complex numbers  $\alpha_1$  and  $\alpha_2$  related to the components of the corresponding unit vectors by

$$\alpha = \frac{a_x + i \cdot a_y}{1 + a_z} . \quad (2.2)$$

In terms of the complex numbers  $\alpha_1$  and  $\alpha_2$ , the state of the system can be specified, ignoring normalization, by

$$|\psi\rangle = |1\rangle + \frac{\alpha_1 + \alpha_2}{\sqrt{2}} |0\rangle + \alpha_1 \cdot \alpha_2 |-1\rangle . \quad (2.3)$$

From (2.3) one can work backwards and determine the two Majorana vectors of the state. This is straightforward and only involves the solution of quadratic equations.

However, if one Majorana vector is (0, 0,-1), a cleverer strategy would have to be used. In this case  $\alpha_1 = \infty$  while  $\alpha_2$  is finite , so that (2.3) can be rewritten as

$$|\psi\rangle = |1\rangle + \frac{\alpha_1}{\sqrt{2}} |0\rangle + \alpha_1 \cdot \alpha_2 |-1\rangle . \quad (2.4)$$

Dividing out by  $\alpha_1$  (which is infinitely large) leads to

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \alpha_2 |-1\rangle . \quad (2.5)$$

The ratio of the coefficients of the second and first terms therefore allows  $\alpha_2$  to be determined. If both alphas are infinite, then  $|\psi\rangle = (0, 0, 1)$ .

For the reverse conversion, the above equations are also used, determining the alpha values first, then the vectors, using the following equations:

$$a_x = \frac{2 \cdot \alpha_R}{1 + |\alpha|^2}; \quad a_y = \frac{2 \cdot \alpha_I}{1 + |\alpha|^2}; \quad a_z = \frac{1 - |\alpha|^2}{1 + |\alpha|^2} . \quad (2.6)$$

Before determining the alpha values, though, the vector must be divided by its first entry to make the first entry 1, thereby allowing (2.3) to be used to determine the two values of alpha. One can solve for the two values of alpha using the following sequence of steps:

$$\Psi_3 = \alpha_1 \cdot \alpha_2, \quad (2.7)$$

$$\Psi_2 = \frac{\frac{\Psi_3}{\alpha} + \alpha}{\sqrt{2}}, \quad (2.8)$$

$$\Psi_2 \cdot \alpha \cdot \sqrt{2} = \Psi_3 + \alpha^2, \quad (2.9)$$

$$\alpha = \frac{\Psi_2 \cdot \sqrt{2} \pm \sqrt{2 \cdot \Psi_2^2 - 4 \cdot \Psi_3}}{2}, \quad (2.10)$$

in which the unsubscripted value for alpha could be either alpha value since interchanging them gives the same equation. Since Eq. (2.10) has two solutions, it gives the solutions for both alphas. Inserting this into the equations for  $\mathbf{a}$ , the values for these vectors can be found. However, this will not work when  $\psi_1 = 0$  since the equation for  $\mathbf{a}$  assumes a certain form of the vector where  $\psi_1 = 1$ .

For the case of  $\psi_1 = 0$ , the method used for  $\mathbf{a} = (0, 0, -1)$  above can be applied in reverse. It is known that one  $\mathbf{a} = (0, 0, -1)$ . Let's call this  $\mathbf{a}_1$ . After dividing  $\mathbf{a}$  by  $\psi_2 \cdot \sqrt{2}$ ,  $\alpha_2$  can be determined using equation 2.5, and  $\mathbf{a}_2$  is found from equation 2.6. If both  $\psi_1$  and  $\psi_2$  are 0, the resulting two vectors are  $(0, 0, -1)$  and  $(0, 0, -1)$ , as can be seen from the fact that these vectors cause the alphas to become infinite.

## **Chapter 3:**

### **The Three Dimensional Systems**

This chapter will explore two different proofs of the Bell-Kochen-Specker (BKS) theorem in three dimensions based on the symmetry of the cube, one given by Peres [4], and the other by Penrose [5], and explain the relationship between them. Peres discovered 33 real rays in three-dimensional space, and Penrose found 33 complex three-dimensional rays, that prove the theorem. First we will list the Peres rays as vectors in real 3-D space and also specify all the orthogonalities between them. Then we will list the Penrose rays by their two Majorana vectors and specify all the orthogonalities between them. After that, it will be shown that the Peres rays can be renumbered so that they become isomorphic to the Penrose rays, in the sense that the orthogonality tables of the two sets become identical. This will then allow the same proof of the BKS theorem to be given for both sets of rays.

#### **3.1 The 33 Peres Rays**

Peres introduced a set of 33 real 3-d rays as the vectors from the center of a cube to 33 points on its surface. These points are shown in Figure 3.1, which is based on the diagram in [8]. The vectors terminate at the midpoints of three faces, the midpoints of six edges, the midpoints of the edges of the inner squares (twelve such) and the vertices of the inner squares (twelve such). Since the negative of a vector is identical to the original vector, only one such vector in each such pair is taken. The points obtained in this way are displayed in Table 3.1.

From Table 3.1, the orthogonalities are determined easily, since two rays are orthogonal if their dot product is 0. The orthogonalities found in this way are listed in Table 3.2. Note that there are 16 triads of mutually orthogonal rays and 24 pairs (“dyads”) of orthogonal rays.

Number	Ray
1	(1, 0, 0)
2	(0, 1, 0)
3	(0, 0, 1)
4	(1, 0, 1)
5	(1, 1, 0)
6	(0, 1, 1)
7	(-1, 1, 0)
8	(-1, 0, 1)
9	(0, -1, 1)
10	(1, 0, $\sqrt{2}$ )
11	( $\sqrt{2}$ , 0, 1)
12	(0, 1, $\sqrt{2}$ )
13	(0, $\sqrt{2}$ , 1)
14	(1, $\sqrt{2}$ , 0)
15	( $\sqrt{2}$ , 1, 0)
16	(-1, 0, $\sqrt{2}$ )
17	( $\sqrt{2}$ , 0, -1)
18	(0, -1, $\sqrt{2}$ )
19	(0, $\sqrt{2}$ , -1)
20	(-1, $\sqrt{2}$ , 0)
21	( $\sqrt{2}$ , -1, 0)
22	(1, 1, $\sqrt{2}$ )
23	(1, $\sqrt{2}$ , 1)
24	( $\sqrt{2}$ , 1, 1)
25	(-1, -1, $\sqrt{2}$ )
26	(-1, $\sqrt{2}$ , -1)
27	( $\sqrt{2}$ , -1, -1)
28	(-1, 1, $\sqrt{2}$ )
29	(-1, $\sqrt{2}$ , 1)
30	( $\sqrt{2}$ , -1, 1)
31	(1, -1, $\sqrt{2}$ )
32	(1, $\sqrt{2}$ , -1)
33	( $\sqrt{2}$ , 1, -1)

*Table 3.1: The 33 rays of Peres, determined from Fig. 3.1 in the manner explained in the text.*

*The components of the rays are given in the standard basis.*

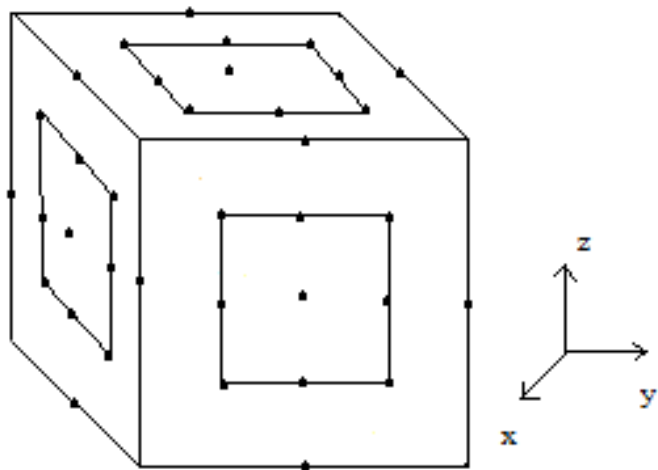


Figure 3.1: Shown are the points on a cube that determine the Peres rays. The center of the cube is at  $(0, 0, 0)$ . Each side is of length 2. The inner squares have sides of length  $\sqrt{2}$  and share the same center as the larger squares.

Triads		Dyads	
1 – 2 – 3	4 – 29 – 32	10 – 27	16 – 24
1 – 6 – 9	5 – 28 – 31	10 – 33	16 – 30
1 – 12 – 19	6 – 30 – 33	11 – 25	17 – 22
1 – 13 – 18	7 – 22 – 25	11 – 28	17 – 31
2 – 4 – 8	8 – 23 – 26	12 – 26	18 – 23
2 – 10 – 17	9 – 24 – 27	12 – 32	18 – 29
2 – 11 – 16		13 – 25	19 – 22
3 – 5 – 7		13 – 31	19 – 28
3 – 14 – 21		14 – 27	20 – 24
3 – 15 – 20		14 – 30	20 – 33
		15 – 26	21 – 23
		15 – 29	21 – 32

Table 3.2: The table shows the orthogonal triads and dyads of the Peres rays listed in Table 3.1.

### 3.2 The 33 Penrose Rays

Penrose introduced another set of 33 rays [5]. These are complex three dimensional rays with pairs of Majorana vectors based on the geometry of a cube. All such Majorana vectors are go from the center of the cube to the points on the cube shown in Figure 2. Described in Penrose’s paper [5], the vectors come in four types of pairs. These pairs are opposite face points (three of them, rays 1-3), opposite edge points (six of them, rays 4-9), double edge points, where the same edge point is taken twice (twelve of them, rays 10-21), and pairs of edge points that are opposite across a face, (twelve of them, rays 22-33). The Majorana vectors for the Penrose rays are presented in Table 3.3.

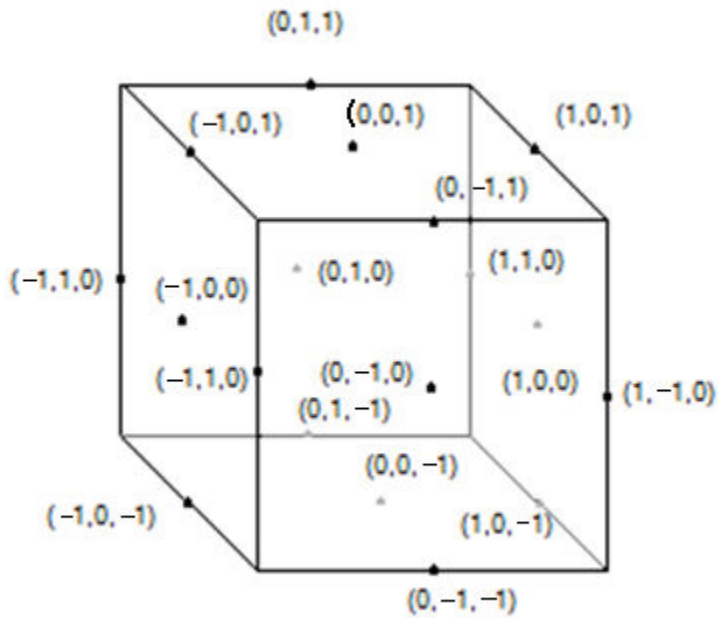


Figure 3.2: Shown are the points on a cube that determine the Penrose rays.

The orthogonalities of the Penrose rays can be determined in one of two ways. The numerical method [9] requires that for two three dimensional rays with Majorana vectors  $(a_1, a_2)$  and  $(b_1, b_2)$  to be orthogonal, the following condition must be satisfied:

$$2(1 + a_1 \bullet b_1)(1 + a_2 \bullet b_2) + 2(1 + a_1 \bullet b_2)(1 + a_2 \bullet b_1) = (a_1 \bullet a_2 - 1)(b_1 \bullet b_2 - 1). \quad (3.1)$$

Penrose also presents a geometric method for determining orthogonalities in his set of 33 rays [5]. This method states that  $(a_1, a_2)$  forms a triad with  $(b_1, b_2)$  and  $(c_1, c_2)$  if  $a_1$  and  $a_2$  are antipodal, the lines between  $b_1$  and  $b_2$ , and  $c_1$  and  $c_2$  are both perpendicular to one another and to the line one between  $a_1$  and  $a_2$ , and the distance from  $a_1$  to  $b_1b_2$  is equal to that from  $a_2$  to  $c_1c_2$ . For the dyads, a ray with both Majorana vectors identical  $(a, a)$  is orthogonal to any pair of points where one of the two points is antipodal to  $a$ .

Using either of these methods, the orthogonalities between all the Penrose rays can be found to be those in Table 4. Both methods were used as a check and both gave the same results. As can be seen, as in the Peres case there are 16 triads and 24 dyads.

Number	Rays
1	(1, 0, 0); (-1, 0, 0)
2	(0, 1, 0); (0, -1, 0)
3	(0, 0, 1); (0, 0, -1)
4	(0, 1, 1); (0, -1, -1)
5	(0, -1, 1); (0, 1, -1)
6	(1, 0, 1); (-1, 0, -1)
7	(-1, 0, 1); (1, 0, -1)
8	(1, 1, 0); (-1, -1, 0)
9	(-1, 1, 0); (1, -1, 0)
10	(0, 1, 1); (0, 1, 1)
11	(0, -1, 1); (0, -1, 1)
12	(0, 1, -1); (0, 1, -1)
13	(0,-1,-1); (0,-1,-1)
14	(1, 0, 1); (1, 0, 1)
15	(-1, 0, 1); (-1, 0, 1)
16	(1, 0, -1); (1, 0, -1)
17	(-1,0,-1); (-1,0,-1)
18	(1, 1, 0); (1, 1, 0)
19	(-1, 1, 0); (-1, 1, 0)
20	(1, -1, 0); (1, -1, 0)
21	(-1,-1,0); (-1,-1,0)
22	(0, 1, 1); (0, -1, 1)
23	(0, 1, 1); (0, 1, -1)
24	(0,-1,-1); (0, -1, 1)
25	(0,-1,-1); (0, 1, -1)
26	(1, 0, 1); (-1, 0, 1)
27	(1, 0, 1); (1, 0, -1)
28	(-1,0,-1); (-1, 0, 1)
29	(-1,0,-1); (1, 0, -1)
30	(1, 1, 0); (-1, 1, 0)
31	(1, 1, 0); (1, -1, 0)
32	(-1,-1,0); (-1, 1, 0)
33	(-1,-1,0); (1, -1, 0)

*Table 3.3: The table lists the rays representing the Penrose set. The Penrose set is listed in terms of Majorana vectors, and the method to determine them is described in this section using Figure 3.2.*

Triads		Dyads	
1 – 2 – 3	4 – 10 – 13	10 – 24	16 – 26
1 – 4 – 5	5 – 11 – 12	10 – 25	16 – 28
1 – 26 – 33	6 – 14 – 17	11 – 25	17 – 26
1 – 29 – 30	7 – 15 – 16	11 – 23	17 – 27
2 – 6 – 7	8 – 18 – 21	12 – 22	18 – 32
2 – 22 – 32	9 – 19 – 20	12 – 24	18 – 33
2 – 25 – 31		13 – 22	19 – 31
3 – 8 – 9		13 – 23	19 – 33
3 – 23 – 28		14 – 28	20 – 30
3 – 24 – 27		14 – 29	20 – 32
		15 – 27	21 – 30
		15 – 29	21 – 31

*Table 3.4: The table shows the triads and dyads of the Penrose rays.*

### 3.3 Isomorphism Between the Peres and Penrose Rays

It can be observed from Tables 3.2 and 3.4 that the orthogonalities of the two sets are similar. They both have four different types of rays, and each group of rays has the same number of members. Each group of rays has the same types of orthogonalities as well. This raises the possibility that the two sets of rays are isomorphic to one another, i.e., it is possible to set up a one-to-one correspondence between the rays in such a way that the orthogonality table for one goes over into that of the other. It was determined that this can be done despite the fact that the two sets of rays are not rotations of each other as Penrose points out [5]. Appendix 1 shows the method used to determine both that there is a mapping and to construct one such mapping. The mapping thus found is displayed in Table 3.5.

Using Table 3.5, the rays can be renumbered so that equivalent rays have the same number, allowing the orthogonality tables, proof tree and criticality proof to be identical for both.

<b>Penrose:</b>	<b>Peres:</b>
4	6
5	9
6	4
7	8
8	7
9	5

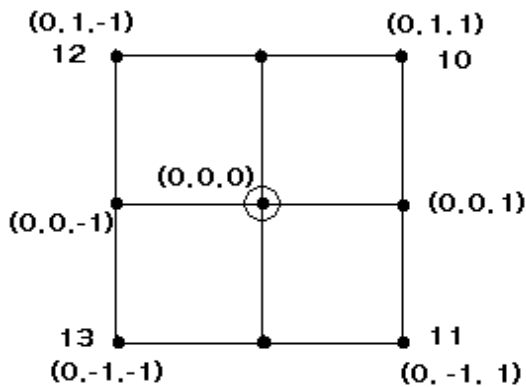
10	30
11	24
12	27
13	33
14	29
15	23
16	26
17	32
18	22
19	28
20	31
21	25
22	10
23	20
24	14
25	16
26	12
27	21
28	15
29	18
30	13
31	11
32	17
33	19

*Table 3.5: The table shows an isomorphism between the Peres and Penrose sets of rays. The rays 1-3 of the Peres set correspond to 1-3 of the Penrose set and are not included in this table.*



freedom in this choice, thereby allowing the argument to prove no-colorability.

This proof can be seen to be sufficient using the symmetry of the Penrose cube. We begin by observing that if a valid coloring exists, one of the rays in the triad consisting of rays 1, 2 and 3 must be chosen green. We can take this to be ray 1 because the rotation indicated in brackets after this triad (and its square) can be used to exchange ray 1 with either 2 or 3 while leaving the entire set of rays invariant (and their mutual orthogonalities unaffected). With the first ray chosen, a choice of 10 and 11, 10 and 12, 13 and 11, and 13 and 12 are all symmetric in the same way. Since 4 and 5 are both red after choosing 1, either 10 or 13 must be green and 11 or 12 must be green and both qualifications are independent of each other. Figure 3.3 shows a cross-section of the cube for the Penrose rays that can be used to see this symmetry. As can be seen, rotating the cube around the line created by the first vector (seen vertically through the origin) will cause one set of two vectors to map onto another, while retaining the first vector and the shape of the cube and cubically symmetric orthogonality table.



*Figure 3.3: A cross-section of the cube used to determine the Penrose rays, showing the symmetries that are used to reduce the length of the proof of non-criticality.*

### 3.5 Criticality of the 33 Rays

A set of rays that can be shown to prove the BKS theorem with a no-coloring proof is said to be “critical” if deleting even a single ray from the set causes the remaining rays to become colorable. The 33 Penrose (and Peres) rays form a critical set. This is demonstrated in Table 3.6, which shows how a satisfactory coloring can be achieved if the ray indicated in the first column is deleted. Following the deleted ray are listed all the rays that are to be colored green (with all the remaining rays to be colored red). Nine green rays are needed when rays 1-3 are deleted, but 10 green rays are needed in all the remaining cases.

### 3.6 The Rays and Orthogonalities of the Peres and Penrose Sets in Both Standard Form and as Majorana Vectors

Listed in Table 3.7 are the Penrose rays and corresponding Peres rays in both standard and Majorana notation with

$$\begin{aligned}
 A &= 2 - \sqrt{2} & E &= \frac{i}{2 + \sqrt{2}} = iA/2 \\
 B &= 1 - \sqrt{2} & F &= \frac{i}{2 - \sqrt{2}} \\
 C &= \sqrt{-1 + 2\sqrt{2}} & & \\
 D &= -1 + \sqrt{1 + 2\sqrt{2}} & & 
 \end{aligned} \tag{3.2}$$

The alternative form of each set of rays was worked out using the methods described in Ch.2.

### 3.7 Other Isomorphic Sets

To determine if there are any other sets of rays with the same orthogonality table as the Peres and Penrose sets, the orthogonalities were taken as the starting point and used to determine the rays that fit them, with an arbitrary rotation of the entire set and the normalization constant of the rays left as free parameters. Appendix 2 presents the method used to achieve this goal. The result is presented in Table 3.8. All real results are equivalent to the Peres set, but the one with in Table 3.7 corresponds to a  $a = 1, b=1, d = \sqrt{2}$ .

The Penrose set however is not in this form. The first three rays don't form a basis for the set. To put the Penrose rays in this form, the entire set must be put into the form where all rays are in the basis of

the first three. Using Maple to rotate the Penrose real rays by  $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$  and multiplying

each ray by a phase to adjust the phase to fit the form in Table 3.8, it was determined that the Penrose case corresponds to  $a = -i, b = -1, d = -\sqrt{2}$ .

In determining other potential sets, the number of isomorphisms possible between the two sets was determined as well. Since there are three parameters that can be positive or negative, it can be seen that the Peres set has 48 automorphisms, 8 for each combination of the first 3 rays. Thus there were 48 different ways to construct an isomorphism between the Peres and Penrose sets.

<b>Ray</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>
1	2	4	8	12	14	16	19	23	27	
2	1	6	8	10	12	16	19	23	27	
3	1	6	8	10	11	15	20	22	31	
4	1	6	8	11	15	20	22	24	28	31
5	1	6	8	10	16	20	22	23	27	31
6	1	7	8	10	11	20	22	27	28	31
7	1	6	8	10	11	20	22	27	28	31
8	1	6	9	10	12	16	23	27	31	32
9	1	6	8	10	12	16	23	27	31	32
10	1	6	8	11	15	20	22	24	28	31
11	1	6	8	10	16	20	22	24	27	31
12	1	6	8	10	16	20	22	23	27	31
13	1	6	8	11	15	20	22	23	28	31
14	1	7	8	10	11	20	22	24	28	31
15	1	6	8	10	11	20	22	27	28	31
16	1	6	8	10	11	20	22	27	28	31
17	1	7	8	10	11	20	22	27	28	31
18	1	6	9	10	12	16	23	27	31	32
19	1	6	8	10	12	16	23	27	31	32
20	1	6	8	10	12	16	23	27	31	32
21	1	6	9	10	12	16	23	27	31	32
22	1	6	8	10	12	16	20	23	27	31
23	1	6	8	10	11	16	20	22	27	31
24	1	6	8	10	11	15	20	22	28	31
25	1	6	8	10	12	16	19	23	27	32
26	2	4	8	12	14	16	19	23	27	30
27	1	6	8	10	11	15	20	22	28	31
28	1	6	8	10	11	16	20	22	27	31
29	2	4	8	12	14	16	20	23	27	33
30	2	4	8	12	14	16	20	23	27	33
31	1	6	8	10	12	16	19	23	27	32
32	1	6	8	10	12	16	20	23	27	31
33	2	4	8	12	14	16	19	23	27	30

*Table 3.6: Displayed are possible sets of green rays when the ray on the left is removed from the set. All other rays would be colored red, showing that the set can be colored.*

	Peres		Penrose	
	<i>Standard</i>	<i>Majorana</i>	<i>Standard</i>	<i>Majorana</i>
1	(1, 0, 0)	(0, 0, 1); (0, 0, 1)	(1, 0, -1)	(1, 0, 0); (-1, 0, 0)
2	(0, 1, 0)	(0, 0, 1); (0, 0, -1)	(1, 0, 1)	(0, 1, 0); (0, -1, 0)
3	(0, 0, 1)	(0, 0, -1); (0, 0, -1)	(0, 1, 0)	(0, 0, 1); (0, 0, -1)
4	(0, 1, 1)	(0,0,-1); ( $2^{3/2}$ ,0,1)	(1, -i√2, 1)	(0, 1, 1); (0, -1, -1)
5	(0, -1, 1)	(0,0,-1); ( $-2^{3/2}$ ,0,1)	(1, i√2, 1)	(0, -1, 1); (0, 1, -1)
6	(1, 0, 1)	(0, 1, 0); (0, -1, 0)	(1, -√2, -1)	(1, 0, 1); (-1, 0, -1)
7	(-1, 0, 1)	(1, 0, 0); (-1, 0, 0)	(1, √2, -1)	(-1, 0, 1); (1, 0, -1)
8	(-1, 1, 0)	(0,0,1); ( $-2^{3/2}$ ,0,-1)	(1, 0, -i)	(1, 1, 0); (-1, -1, 0)
9	(1, 1, 0)	(0,0,1); ( $2^{3/2}$ ,0,-1)	(1, 0, i)	(-1, 1, 0); (1, -1, 0)
10	(√2, -1, 1)	(-2, -2C, A); (-2, 2C, A)	(1, 2E, 2E <sup>2</sup> )	(0, 1, 1); (0, 1, 1)
11	(√2, 1, 1)	(2, -2C, A); (2, 2C, A)	(1, 2F, 2F <sup>2</sup> )	(0, -1, 1); (0, -1, 1)
12	(√2, -1, -1)	(-2D, 0, D + 2 <sup>1/2</sup> ); (-2D - 4, 0, -D - 2 <sup>1/2</sup> )	(1, -2E, 2E <sup>2</sup> )	(0, 1, -1); (0, 1, -1)
13	(√2, 1, -1)	(2D, 0, -D - 2 <sup>1/2</sup> ); (2D + 4, 0, D + 2 <sup>1/2</sup> )	(1, -2F, 2F <sup>2</sup> )	(0,-1,-1); (0,-1,-1)
14	(-1, √2, 1)	(B, 0, 2 <sup>1/2</sup> ); (B, 0, -2 <sup>1/2</sup> )	(1, A, A <sup>2</sup> /2)	(1, 0, 1); (1, 0, 1)
15	(1, √2, 1)	(1,0,0); (1,0,0)	(1, -2iF, -2F <sup>2</sup> )	(-1, 0, 1); (-1, 0, 1)
16	(-1, √2, -1)	(-1,0,0); (-1,0,0)	(1, -A, A <sup>2</sup> /2)	(1, 0, -1); (1, 0, -1)
17	(1, √2, -1)	(1+2 <sup>1/2</sup> ,0, -2 <sup>1/2</sup> ); (-B,0, 2 <sup>1/2</sup> )	(1, 2iF, -2F <sup>2</sup> )	(-1,0,-1); (-1,0,-1)
18	(1, 1, √2)	(2 <sup>1/2</sup> , 2 <sup>1/2</sup> *C, B); 2 <sup>1/2</sup> , -2 <sup>1/2</sup> *C, B)	(1, 1 + i, i)	(1, 1, 0); (1, 1, 0)
19	(-1, 1, √2)	(-2 <sup>3/2</sup> *(D+2), 0, - 2*(D+2 <sup>3/2</sup> )); (2 <sup>3/2</sup> *D, 0, -2*(D+B))	(1, 1 - i, -i)	(-1, 1, 0); (-1, 1, 0)
20	(1, -1, √2)	(-2 <sup>1/2</sup> , 2 <sup>1/2</sup> *C, B); -2 <sup>1/2</sup> , -2 <sup>1/2</sup> *C, B)	(1, -1 + i, -i)	(1, -1, 0); (1, -1, 0)
21	(-1, -1, √2)	(2 <sup>3/2</sup> *(D+2), 0, - 2*(D+2 <sup>3/2</sup> )); (-2 <sup>3/2</sup> *D, 0, -2*(D+B))	(1, -1 - i, i)	(-1,-1,0); (-1,-1,0)

22	$(1, 0, \sqrt{2})$	$(0, 2^{5/4}, B);$ $(0, -2^{5/4}, B)$	$(1, 2i, -1)$	$(0, 1, 1); (0, -1, 1)$
23	$(-1, \sqrt{2}, 0)$	$(0, 0, 1);$ $(-4, 0, -3)$	$(1, 0, A^2/2)$	$(0, 1, 1); (0, 1, -1)$
24	$(1, \sqrt{2}, 0)$	$(4, 0, -3);$ $(0, 0, 1)$	$(1, 0, -2F^2)$	$(0,-1,-1); (0, -1, 1)$
25	$(-1, 0, \sqrt{2})$	$(2^{5/4}, 0, B);$ $(-2^{5/4}, 0, B)$	$(1, -2i, -1)$	$(0,-1,-1); (0, 1, -1)$
26	$(0, 1, \sqrt{2})$	$(0,0,-1); (1,0,0)$	$(1, 2, 1)$	$(1, 0, 1); (-1, 0, 1)$
27	$(\sqrt{2}, -1, 0)$	$(0, 0, 1); (-1, 0, 0)$	$(1, 0, -A^2/2)$	$(1, 0, 1); (1, 0, -1)$
28	$(\sqrt{2}, 1, 0)$	$(1, 0, 0); (0, 0, 1)$	$(1, 0, 2F^2)$	$(-1,0,-1); (-1, 0, 1)$
29	$(0, -1, \sqrt{2})$	$(0,0,-1), (-1,0,0)$	$(1, -2, 1)$	$(-1,0,-1); (1, 0, -1)$
30	$(0, \sqrt{2}, 1)$	$(4,0,3), (0,0,-1)$	$(1, 1, 1)$	$(1, 1, 0); (-1, 1, 0)$
31	$(\sqrt{2}, 0, 1)$	$(0, 2^{7/4}, A);$ $(0, -2^{7/4}, A)$	$(1, i, -1)$	$(1, 1, 0); (1, -1, 0)$
32	$(\sqrt{2}, 0, -1)$	$(2^{7/4}, 0, A);$ $(-2^{7/4}, 0, A)$	$(1, -i, -1)$	$(-1,-1,0); (-1, 1, 0)$
33	$(0, \sqrt{2}, -1)$	$(-4,0,3); (0,0,-1)$	$(1, -1, 1)$	$(-1,-1,0); (1, -1, 0)$

Triads		Pairs	
1 – 2 – 3	4 – 10 – 13	10 – 24	16 – 26
1 – 4 – 5	5 – 11 – 12	10 – 25	16 – 28
1 – 26 – 33	6 – 14 – 17	11 – 25	17 – 26
1 – 29 – 30	7 – 15 – 16	11 – 23	17 – 27
2 – 6 – 7	8 – 18 – 21	12 – 22	18 – 32
2 – 22 – 32	9 – 19 – 20	12 – 24	18 – 33
2 – 25 – 31		13 – 22	19 – 31
3 – 8 – 9		13 – 23	19 – 33
3 – 23 – 28		14 – 28	20 – 30
3 – 24 – 27		14 – 29	20 – 32
		15 – 27	21 – 30
		15 – 29	21 – 31

Table 3.7: The isomorphic Peres and Penrose vectors in both the standard and Majorana vector form are present in the first table. The second table shows the combine orthogonality table. Equation 2 shows the value for A, B, C, D, E, and F.

4	( 0, 1, a)	22	( 1, 0, ad)
5	( 0, a*, -1)	23	( 1, -d, 0)
6	( 1, 0, b)	24	( 1, d, 0)
7	( b*, 0, -1)	25	( 1, 0, -ad)
8	( 1, c, 0)	26	( 0, 1, bd*)
9	( c*, -1, 0)	27	( -d*, 1, 0)
10	( a*d*, -a*, 1)	28	( d*, 1, 0)
11	( d*, 1, a)	29	( 0, 1, -bd*)
12	( -d*, 1, a)	30	( 0, b*d, 1)
13	(-a*d*, -a*, 1)	31	( a*d*, 0, 1)
14	( -b*, b*d, 1)	32	(-a*d*, 0, 1)
15	( 1, d, b)	33	( 0, -b*d, 1)
16	( 1, -d, b)		
17	( -b*, -b*d, 1)		
18	( -c*, 1, bd*)		
19	( 1, c, -ad)		
20	( 1, c, ad)		
21	( -c*, 1, -bd*)		

Table 3.8: The arbitrary rays determined by all triads all the pairs with  $|a|^2 = 1, |b|^2 = 1, |d|^2 = 2,$

and  $c = -ab^* \frac{d}{d^*}.$

## **Chapter 4: Conclusion**

In exploring the three dimensional systems of 33 rays given by Peres and Penrose, it was found that both sets can be renumbered so that they share a common orthogonality table. Continuing from this, a general family of 33 rays was found with this orthogonality table that contains both of the Peres and Penrose sets as special cases. This general family of rays is characterized by three complex parameters of fixed modulus but continually varying phase. Since all these sets of rays share the same orthogonality table, the no-colorability proof and criticality proof are identical for all.

Both the Peres and Penrose sets are based on the symmetry of the cube. Using the symmetry of the Penrose cube, a no-colorability proof was found that is shorter than the one given by Peres for his set or rays.

Potential future research includes examining the symmetry of the general set to determine if it possesses cubic symmetry in the Majorana vector form and looking for general sets based on the orthogonality tables of sets of rays not isomorphic to this one.

## References

1. J.S.Bell, "On the Einstein-Podolsky-Rosen paradox", *Physics I*, 195-200 (1964). Reprinted in J.S.Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, New York, 1987).
2. J.S.Bell, "On the problem of hidden variables in quantum mechanics", *Rev.Mod.Phys.* **38**, 447-52 (1966).
3. S.Kochen and E.P.Specker, "The problem of hidden variables in quantum mechanics", *J. Math. Mech.* **17**, 59-88 (1967).
4. A.Peres, "Two simple proofs of the Kochen-Specker theorem", *J.Phys.* **A24**, 174-8 (1991).
5. R.Penrose, "On Bell non-locality without probabilities: some curious geometry", in *Quantum Reflections* J.Ellis and D.Amati, Eds, Cambridge University Press, New York (2000).
6. See A.Peres, *Quantum Theory: Concepts and Methods*, Kluwer Academic Publishers, Boston (1995), Ch.7.
7. For a description of the Majorana treatment of spin see R.Penrose, *The emperor's new mind: concerning computers, minds and the laws of physics*, Oxford University Press, 1989.
8. F. Lee-Elkin, "Non-Coloring Proofs of the Kochen-Specker Theorem", WPI MQP report PH-PKA-FL98 (1998).
9. P. K. Aravind, unpublished notes.

## Appendix 1:

### Determining the Isomorphism between the Peres and Penrose Sets

This Appendix shows how to set up a correspondence, or mapping, between the Peres and Penrose rays so that the orthogonalities in Table 3.2 go over into those in Table 3.4. In determining the mapping of one set to the other, rays 1, 2 and 3 are chosen to remain the same. The next 6 rays must be permuted in a way to be determined later. The mapping starts with rays 10-33 from both sets. The first group of 12 rays from one set must map to the second group of 12 in the other set. This mapping must preserve the pairs as well as the triads. The orthogonalities for rays 10 – 33 of both sets are presented in Table A1.1, as determined from Tables 3.2 and 3.4.

Peres			Penrose		
12 – 19	10 – 27	16 – 24	10 – 13	10 – 24	16 – 26
13 – 18	10 – 33	16 – 30	11 – 12	10 – 25	16 – 28
10 – 17	11 – 25	17 – 22	14 – 17	11 – 25	17 – 26
11 – 16	11 – 28	17 – 31	15 – 16	11 – 23	17 – 27
14 – 21	12 – 26	18 – 23	18 – 21	12 – 22	18 – 32
15 – 20	12 – 32	18 – 29	19 – 20	12 – 24	18 – 33
29 – 32	13 – 25	19 – 22	26 – 33	13 – 22	19 – 31
28 – 31	13 – 31	19 – 28	29 – 30	13 – 23	19 – 33
30 – 33	14 – 27	20 – 24	22 – 32	14 – 28	20 – 30
22 – 25	14 – 30	20 – 33	25 – 31	14 – 29	20 – 32
23 – 26	15 – 26	21 – 23	23 – 28	15 – 27	21 – 30
24 – 27	15 – 29	21 – 32	24 – 27	15 – 29	21 – 31

*Table A1.1: The composite list of all pairs containing only rays 10 – 33. This combines such information from both the orthogonal triads and orthogonal pairs.*

We use the orthogonalities in Table A1.1 to create a table of potential mappings that allow for only a small number of possibilities with  $1 = 1$ ,  $2 = 2$ , and  $3 = 3$ . These mappings are shown in Table A1.2.

<b>Peres</b>	<b>Possible Penrose</b>	<b>Peres</b>	<b>Possible Penrose</b>
10	22, 32, 25, 31	22	18, 19, 20, 21
11	22, 32, 25, 31	23	14, 15, 16, 17
12	26, 33, 29, 30	24	10, 11, 12, 13
13	26, 33, 29, 30	25	18, 19, 20, 21
14	23, 28, 24, 27	26	14, 15, 16, 17
15	23, 28, 24, 27	27	10, 11, 12, 13
16	22, 32, 25, 31	28	18, 19, 20, 21
17	22, 32, 25, 31	29	14, 15, 16, 17
18	26, 33, 29, 30	30	10, 11, 12, 13
19	26, 33, 29, 30	31	18, 19, 20, 21
20	23, 28, 24, 27	32	14, 15, 16, 17
21	23, 28, 24, 27	33	10, 11, 12, 13

*Table A1.2: Potential Penrose rays identical in the orthogonalities to the listed Peres rays.*

*Listed are the rays 10 – 33. Ray 1 corresponds to ray 1, 2 to 2, and 3 to 3. Peres ray 4 or 8 corresponds to Penrose ray 6 or 7, 5 or 7 to 8 or 9, and 6 or 9 to 4 or 5.*

The next step is to choose some Penrose rays to correspond to the Peres rays, and find from the orthogonalities what the remaining rays must correspond to. This is repeated until either a final mapping is determined or it is found that no mapping exists. Table A1.3 is determined by choosing Peres 10 to correspond to Penrose 22. Table A1.4 chooses Peres 22 = Penrose 18 in addition and Table A1.5 further chooses Peres 4 = Penrose 6. This isomorphism was then confirmed by replacing the old Penrose ray numbers with the new ones and determining if the resulting orthogonalities matched the Peres orthogonalities.

<b>Peres:</b>	<b>Possible Penrose:</b>	<b>Peres:</b>	<b>Possible Penrose:</b>
4	6, 7	7	8, 9
5	8, 9	8	6, 7
6	4, 5	9	4, 5

10	22	22	18, 20
11	31	23	14, 15, 16, 17
12	26, 29	24	10, 11
13	30, 33	25	19, 21
14	23, 24	26	14, 15, 16, 17
15	27, 28	27	12, 13
16	25	28	19, 21
17	32	29	14, 15, 16, 17
18	26, 29	30	10, 11
19	30, 33	31	18, 20
20	23, 24	32	14, 15, 16, 17
21	27, 28	33	12, 13

*Table A1.3: Potential mappings with Peres 10 equal to Penrose 22.*

<b>Peres:</b>	<b>Possible Penrose:</b>	<b>Peres:</b>	<b>Possible Penrose:</b>
4	6, 7	7	8
5	9	8	7,6
6	4	9	5

10	22	22	18
11	31	23	14, 15
12	26	24	11
13	30	25	21
14	24	26	16, 17
15	28	27	12
16	25	28	19
17	32	29	14, 15
18	29	30	10
19	33	31	20
20	23	32	16, 17
21	27	33	13

*Table A1.4: Potential mappings with Peres 10 equal to Penrose 22 and Peres 22 equal to*

*Penrose 18.*

<b>Peres:</b>	<b>Possible Penrose:</b>	<b>Peres:</b>	<b>Possible Penrose:</b>
4	6	7	8
5	9	8	7
6	4	9	5
10	22	22	18
11	31	23	15
12	26	24	11
13	30	25	21
14	24	26	16
15	28	27	12
16	25	28	19
17	32	29	14
18	29	30	10
19	33	31	20
20	23	32	17
21	27	33	13

*Table A1.5: A potential isomorphism of the Peres and Penrose sets with Peres 10 equal to Penrose 22, Peres 22 equal to Penrose 18, and Peres 4 equal to Penrose 6.*

## Appendix 2:

### Determining the General Set of Rays that Match the Peres and Penrose

#### Set of Orthogonalities

Here we solve the problem of determining the general set of rays that satisfy all the orthogonalities shown in Table 3.4 or 3.7. First, rays 1, 2, and 3 can be taken to be the basis of the set, i.e. we take  $1 = (1,0,0)$ ,  $2 = (0,1,0)$  and  $3 = (0,0,1)$ . Values were then assigned to all rays orthogonal to these three, using the triads to determine which these rays are. In addition, since an overall phase and normalization are unimportant, each ray can have one of its three components set to an arbitrary value without loss of generality and without altering the orthogonality relations. The result is shown in Table A2.1, which contains 9 unknown parameters.

4	$(0, 1, a)$	22	$(1, 0, f)$
5	$(0, a^*, -1)$	23	$(1, h, 0)$
6	$(1, 0, b)$	24	$(1, z, 0)$
7	$(b^*, 0, -1)$	25	$(1, 0, g)$
8	$(1, c, 0)$	26	$(0, 1, d)$
9	$(c^*, -1, 0)$	27	$(-z^*, 1, 0)$
		28	$(-h^*, 1, 0)$
		29	$(0, 1, e)$
		30	$(0, -e^*, 1)$
		31	$(-g^*, 0, 1)$
		32	$(-f^*, 0, 1)$
		33	$(0, -d^*, 1)$

*Table A2.1: Present are the arbitrary rays determined by only the triads containing rays 1, 2, and 3. There are no restrictions on the parameters yet.*

The missing rays were added in using the remaining triads. The results are shown in Table A2.2 with  $A = |a|^2 + 1$   $B = |b|^2 + 1$   $C = |c|^2 + 1$ . This table contains 15 unknown parameters.

4	(0, 1, a)	22	(1, 0, f)
5	(0, a*, -1)	23	(1, h, 0)
6	(1, 0, b)	24	(1, z, 0)
7	(b*, 0, -1)	25	(1, 0, g)
8	(1, c, 0)	26	(0, 1, d)
9	(c*, -1, 0)	27	(-z*, 1, 0)
10	(j, -a*, 1)	28	(-h*, 1, 0)
11	(k, 1, a)	29	(0, 1, e)
12	(A/k*, 1, a)	30	(0, -e*, 1)
13	(A/j*, -a*, 1)	31	(-g*, 0, 1)
14	(-b*, l, 1)	32	(-f*, 0, 1)
15	(1, m, b)	33	(0, -d*, 1)
16	(1, B/m*, b)		
17	(-b*, B/l*, 1)		
18	(-c*, 1, n)		
19	(1, c, p)		
20	(1, c, C/p*)		
21	(-c*, 1, C/n*)		

*Table A2.2: Present are the rays determined by all triads, but only triads. In this table,  $A=|a|^2+1$   $B=|b|^2+1$   $C=|c|^2+1$ . There are no restrictions on the parameters yet.*

To determine the limits on the parameters, the orthogonal pairs were taken, in groups of eight at a time. Starting with rays 10-13 and 22-25, the orthogonalities used to restrict the unknown parameters are as follows:

$$0 = \langle 10|24 \rangle = j^* - za, 0 = \langle 10|25 \rangle = j^* + g, 0 = \langle 11|23 \rangle = k^* + h, 0 = \langle 11|25 \rangle = k^* + a^* g,$$

$$0 = \langle 12|22 \rangle = -\frac{|a|^2+1}{k} + a^* f, 0 = \langle 12|24 \rangle = -\frac{|a|^2+1}{k} + z,$$

$$0 = \langle 13|22 \rangle = -\frac{|a|^2+1}{j} + f, 0 = \langle 13|23 \rangle = -\frac{|a|^2+1}{j} - a^* h.$$

These equations can be simplified to

$$j^* = za = -g, k^* = -h = -a^* g, z = \frac{|a|^2+1}{k} = a^* f, f = \frac{|a|^2+1}{j} = -a^* h,$$

and solved to give

$$|a|^2 = 1, |z|^2 = 2; f = z/a^*, g = -za, h = -z, j = z^* a^*, k = z^*.$$

This leads to the results in Table A2.3 with  $|a|^2 = 1, |z|^2 = 2$ . Highlighted are the relevant rays for the above analysis. These are the rays that are altered due to this analysis.

4	(0, 1, a)	22	(1, 0, az)
5	(0, a*, -1)	23	(1, -z, 0)
6	(1, 0, b)	24	(1, z, 0)
7	(b*, 0, -1)	25	(1, 0, -az)
8	(1, c, 0)	26	(0, 1, d)
9	(c*, -1, 0)	27	(-z*, 1, 0)
10	(a*z*, -a*, 1)	28	(z*, 1, 0)
11	(z*, 1, a)	29	(0, 1, e)
12	(-z*, 1, a)	30	(0, -e*, 1)
13	(-a*z*, -a*, 1)	31	(a*z*, 0, 1)
14	(-b*, l, 1)	32	(-a*z*, 0, 1)
15	(1, m, b)	33	(0, -d*, 1)
16	(1, B/m*, b)		
17	(-b*, B/l*, 1)		
18	(-c*, 1, n)		
19	(1, c, p)		
20	(1, c, C/p*)		
21	(-c*, 1, C/n*)		

*Table A2.3: The arbitrary rays determined by all triads and the first eight pairs with*

$|a|^2 = 1, |z|^2 = 2$ . It is identical to Table A2.2 except where highlighted.

Using the same process as above for rays 14-17 and 26-29, the following equations limit the values of the next set of unknowns:  $|b|^2 = 1; d = bz^*, e = -z^*/b^*, l = z/b, m = z$ .

Table A2.3 now becomes as shown in Table A2.4, with  $|a|^2 = 1, |z|^2 = 2$ , and  $|b|^2 = 1$ , again highlighting the changed rays.

4	(0, 1, a)	22	(1, 0, az)
5	(0, a*, -1)	23	(1, -z, 0)
6	(1, 0, b)	24	(1, z, 0)
7	(b*, 0, -1)	25	(1, 0, -az)
8	(1, c, 0)	26	(0, 1, bz*)
9	(c*, -1, 0)	27	(-z*, 1, 0)
10	(a*z*, -a*, 1)	28	(z*, 1, 0)
11	(z*, 1, a)	29	(0, 1, -bz*)
12	(-z*, 1, a)	30	(0, b*z, 1)
13	(-a*z*, -a*, 1)	31	(a*z*, 0, 1)
14	(-b*, b*z, 1)	32	(-a*z*, 0, 1)
15	(1, z, b)	33	(0, -b*z, 1)
16	(1, -z, b)		
17	(-b*, -b*z, 1)		
18	(-c*, 1, n)		
19	(1, c, p)		
20	(1, c, C/p*)		
21	(-c*, 1, C/n*)		

Table A2.4: The arbitrary rays determined by all triads and the first 16 pairs with

$|a|^2 = 1, |z|^2 = 2$  and  $|b|^2 = 1$ . It is identical to Table A2.3 except where highlighted.

The following equations limit the values of the unknowns from the last set of eight rays:

$c = -ab^* \frac{z}{z^*}, n = bz^*, p = -za$ . The equation for c comes from  $0 = \langle 18|32 \rangle = cf^* + n^*$ , becoming

$cf^* = -n^*$ , or  $c = -\frac{n^*}{f^*} = -\frac{ab^* z}{z^*}$ . The final result for the general set of rays consistent with the

orthogonalities in Table 3.7 is Table A2.5 with  $|a|^2 = 1, |b|^2 = 1, |z|^2 = 2$  and  $c = -ab^* \frac{z}{z^*}$ .

4	( 0, 1, a)	22	( 1, 0, az)
5	( 0, a*, -1)	23	( 1, -z, 0)
6	( 1, 0, b)	24	( 1, z, 0)
7	( b*, 0, -1)	25	( 1, 0, -az)
8	( 1, c, 0)	26	( 0, 1, bz*)
9	( -c*, -1, 0)	27	( -z*, 1, 0)
10	( a*z*, -a*, 1)	28	( z*, 1, 0)
11	( z*, 1, a)	29	( 0, 1, -bz*)
12	( -z*, 1, a)	30	( 0, b*z, 1)
13	(-a*z*, -a*, 1)	31	( a*z*, 0, 1)
14	( -b*, b*z, 1)	32	(-a*z*, 0, 1)
15	( 1, z, b)	33	( 0, -b*z, 1)
16	( 1, -z, b)		
17	( -b*, -b*z, 1)		
18	( -c*, 1, bz*)		
19	( 1, c, -az)		
20	( 1, c, az)		
21	( -c*, 1, -bz*)		

Table A2.5: The general rays determined by all triads and all the pairs in table 3.4 with

$$|a|^2 = 1, |b|^2 = 1, |z|^2 = 2, \text{ and } c = -ab^* \frac{z}{z^*}.$$