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Analysis on Fractals

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Abstract

In this paper we study the standard Dirichlet form and its associated energy measures and Laplacians on the Sierpinski gasket, and Related fractals constructed by Kigami. We first obtain a pointwise formula for the Kusuoka Laplacian, and then the intersection of its domain and that of the standard Laplacian is shown to be the set of harmonic functions. The standard measure is then shown to be the only one with respect to which the harmonic functions satisfy a certain type of mean value property.

1 Acknowledgment

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2 Introduction

There is a well developed theory of Laplacians on the Sierpinski Gasket (SG). SG represents the prototype of a wide class of fractals known as post critically finite (p.c.f) self-similar fractals. Two approaches to the theory were independently developed: there is the indirect probabilistic approach (see [1],[2]) and the direct discrete approach (see [3],[4]).

Two ingredients are needed to define the Laplacian on SG: the Dirichlet form $\mathcal{E}$ and a regular measure μ that assigns positive values to every non-empty open set O ($\mu(O) > 0$). For a given function u defined on SG, $\mathcal{E}(u,u)$ plays the role of $\int |\nabla u|^2 d\mu$ in the Euclidean setting. Harmonic functions are obtained as minimizers of $\mathcal{E}(u,u)$ over all functions that satisfy some boundary conditions. The idea behind the definition of the Laplacian is the integration by part formula

$$\int_0^1 u''(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx$$

where $v \in C^1((0,1))$ with $v(0) = v(1) = 0$ and $u \in C^2((0,1))$. $\mathcal{E}(u,v)$ will replace the right hand side and $d\mu$ will replace dx. Different choices for the measure $d\mu$ will yield different Laplacian.

Two measures will be of particular interest: the Kusuoka measure which is an energy measure with respect to which all other energy measures are continuous, and the standard measure which is the natural self-similar measure that assigns equal weights to cells of the same level. It is proven in [6] that the Kusuoka measure and standard measure are singular. In this paper, we will show that the domains of their Laplacians intersects on the rather small space of harmonic functions. We will also develop, as in the case of the standard Laplacian, a pointwise formula that will allow us to compute explicitely the pointwise value of the Kusuoka Laplacian of functions over a dense subset of SG.

3 Laplacian on Sierpinski Gasket

Let $V_0 = \{q_0, q_1, q_2\}$ be a set of three distinct points in the plane and consider the contractions $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, i=0, 1, 2, defined by:

$$F_i(x) = (x - q_i)/2 + q_i. \tag{1}$$

The Sierpinski Gasket K is the *unique* non-empty compact set that satisfies the following self-similar identity (see [7])

$$K = \bigcup_{i=0}^{2} F_i(K). \tag{2}$$

Clearly, 2 is not the only self-similar identity that determines K; other self-similar identities for K can be obtained by iteration. Indeed, for each $1 \leq m \leq \infty$ we define the set $W_m := \{w_1 w_2 \cdots w_m : w_i = 0, 1, 2\}$, and we call $w \in W_m$ a *word* of length $|w| = m$. We set $W^* := \bigcup_{1 \leq m \leq \infty} W_m$. For a word w, $w \in W_m$, we define the contractions F_w by

$$F_w := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}. \tag{3}$$

It is then easy to verify that

$$K = \bigcup_{|w|=m} F_w(K). \tag{4}$$

For $w \in W^*$, we set $K_w := F_w(K)$, and we will refer to K_w as a *cell of level* $m = |w|$.

We consider the sequence of *vertices* V_m defined by:

$$V_m = \bigcup_{w \in W_m} F_w(V_0), \tag{5}$$

and we set

$$V^* = \bigcup_{0 \leq m \leq \infty} V_m.$$

We call V_0 the *boundary* of K. The Sierpinski Gasket K can then be approximated by a sequence of graphs Γ_m on the vertices sets V_m which are defined by the the edge relations $\sim_m$:

$$\text{for p, q } \in V_m,\ p \sim_m q \text{ if and only if } p, q \in F_w(V_0) \text{ for some } w \in W_m. \tag{6}$$

We will refer to p as a *neighbor* of q in Γ_m if $p \sim_m q$. One should note that any two cells of level m are either disjoint or they intersect at a point in V_m. Also, every point in $V_m \backslash V_0$ has exactly four neighbors and every point in V_0 has two neighbors.

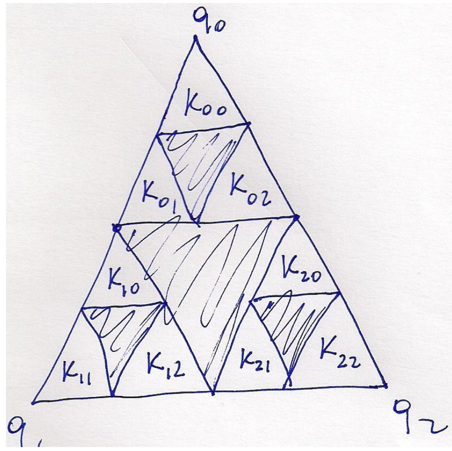

Figure 1: *Cells of level 2*

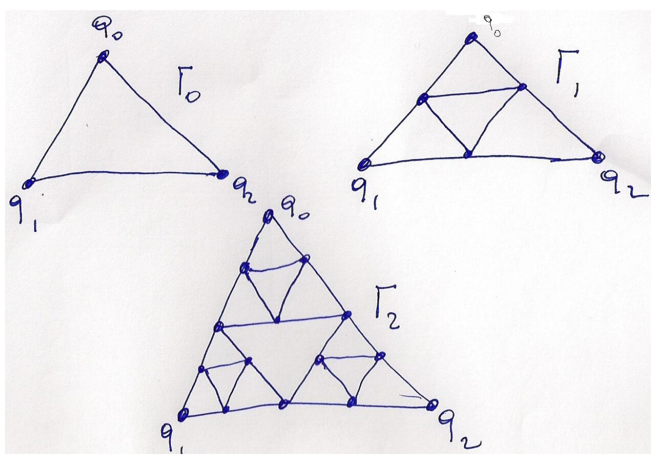

Figure 2: *Graphs* Γ_1, Γ_2 *and* Γ_3

We consider the imbedding of K in the euclidean plane and give K the subspace topology it inherits from $\mathbb{R}^2$. In order to compute integrals of functions defined on K, we need to equip K with a measure. Given μ_i, $i = 0, 1, 2$, three positive constants such that $\sum_{i=0}^{2} \mu_i = 1$, we can construct on K an associated *regular probability measure* μ as follows. We first construct an associated *outer measure* $\tilde{\mu}$ defined for all subsets of K. We assign to a cell K_w of level m the weight $\tilde{\mu}(K_w)$ given by

$$\tilde{\mu}(K_w) =: \mu_w = \prod_{i=0}^{m} \mu_{w_i}. \tag{7}$$

For a general subset C of K, we define $\tilde{\mu}(C)$ by:

$$\tilde{\mu}(C) = \inf\{\sum_{w \in \Delta} \tilde{\mu}(K_w) \mid C \subset \bigcup_{w \in \Delta} K_w,\ w \in W^*\}. \tag{8}$$

Note that when the subset C is an m cell for some integer m, the definition 7 of $\tilde{\mu}(K_w)$ is unambiguous. We can then define an associated measure μ on K by restricting the domain of definition of $\tilde{\mu}$ to the $\sigma - algebra$ $\mathcal{A}$ of measurable sets. It can be easily verified that $\mu(K) = 1$.

All measures on K constructed this way are *self-similar*; they satisfy the self-similar identity

$$\mu(A) = \sum_{i=0}^{2} \mu_i \mu(F_i^{-1}(A)) \tag{9}$$

for all subsets A of K. To prove this, let us first note that for an arbitrary cell K_w we have

$$\mu(F_i K_w) = \mu_i \mu_w = \mu_i \mu(F_w K),$$

which combined with 8 gives

$$\mu(F_i A) = \mu_i \mu(A)$$

for an arbitrary subset A of K. Also,

$$F_i^{-1}(A) = F_i^{-1}(A \cap F_i K).$$

Indeed the inclusion $F_i^{-1}(A) \supset F_i^{-1}(A \cap F_i K)$ holds trivially and $F_i^{-1}(A) \subset F_i^{-1}(A \cap F_i K)$ holds since

$$F_i x \in A \Rightarrow F_i x \in A \cap F_i K.$$

We thus have

$$\begin{aligned}\mu(A) &= \sum_{i=0}^{2} \mu(A \cap F_i K) \\ &= \sum_{i=0}^{2} \mu(F_i F_i^{-1}(A \cap F_i K)) \\ &= \sum_{i=0}^{2} \mu(F_i F_i^{-1}(A)) \\ &= \sum_{i=0}^{2} \mu_i \mu(F_i^{-1}(A)).\end{aligned}$$

It is easy to see that the self-similar identity 9 determines the measure μ uniquely.

The measures we have thus constructed allow us to integrate continuous functions on K. For $f \in C(K)$ we compute its Riemann integral as

$$\int_K f d\mu = \lim_{m\to\infty} \sum_{|w|=m} f(x_w)\mu(F_w K), \tag{10}$$

where x_w is any point belonging to K_w. The fact that K is compact and f is continuous, which makes f absolutely continuous, renders the right hand side of 10 well defined. Making use of the average value of f over the boundary of each cell, we can equivalently compute this integral as

$$\int_K f d\mu = \frac{1}{3} \lim_{m\to\infty} \sum_{i=0}^{2} \sum_{|w|=m} f(F_w q_i)\mu(F_w K). \tag{11}$$

We can transform the self-similar identity for measures 9 into the following self-similar identity for integrals:

$$\int_K f d\mu = \sum_{i=0}^{2} \mu_i \int_K f \circ F_i \; d\mu. \tag{12}$$

Indeed, letting $f = \mathcal{X}_A$, where $\mathcal{X}_A$ denotes the characteristic function of the set A, equation 9 becomes:

$$\int_K f d\mu = \mu(A) = \sum_{i=0}^{2} \mu_i \mu(F_i^{-1}(A)) = \sum_{i=0}^{2} \mu_i \int_K f \circ F_i \; d\mu.$$

Note that we have made use of the fact that $f \circ F_i = \mathcal{X}_{F_i^{-1}(A)}$.

In this paper, we will be mainly interested with the case when $\mu_i = 1/3$, $i = 0, 1, 2$. In this context, the associated probability measure μ is called the standard measure. The standard measure is known to coincide, up to a multiplicative constant, with the Hausdorff measure on K in dimension $\log 3/\log 2$. The exact value of the multiplicative constant is an unsolved problem.

An additional tool that will be of use in our analysis of K is the *Dirichlet form* $\mathcal{E}$. We define $\mathcal{E}$ as a normalized limit of graph energy forms E_m. For each integer m, given two real-valued functions u, v with domain K, $E_m(u, v)$ is defined by:

$$E_m(u, v) = \sum_{p \sim_m q} \left(u(p) - u(q)\right)\left(v(p) - v(q)\right). \tag{13}$$

Note that to compute $E_m(u, v)$ we only need to know the values of the functions u and v on the vertices of the graph Γ_m. Also, the energy forms E_m are bilinear and symmetric. In addition, $E_m(u, u) = 0$ if and only if u is constant (Γ_m's are connected graphs). Therefore, for every integer m, E_m is an inner product on the space of real-valued functions on Γ_m mod the constant functions.

It is easy to verify that the energy forms E_m can be equivalently defined by

$$E_m(u, v) = \sum_{p \in V_m} \sum_{q \sim_m p} u(p)\left(v(p) - v(q)\right). \tag{14}$$

When $u = v$, $E_m(u) := E_m(u, u)$ becomes a quadratic form in u. The relationship between the quadratic form and the bilinear form is expressed by the following polarization identity

$$E_m(u, v) = \frac{1}{4}\left(E_m(u + v) - E_m(u - v)\right). \tag{15}$$

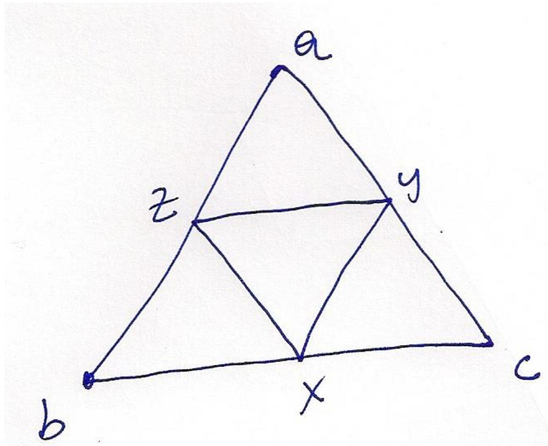

Figure 3: *Harmonic functions*

Given a function u defined on Γ_m, there are many possible ways to extend u to a function defined on Γ_{m+1}. Among all possible extensions of u to Γ_{m+1}, it can easily be shown that there is a unique extension $\tilde{u}$ which minimizes the quadratic form E_{m+1}. In other words, for all functions u' on Γ_{m+1} such that $u'|_{\Gamma_m} = u$, we have $E_{m+1}(u') \geq E_{m+1}(\tilde{u})$. We call this energy-minimizing extension $\tilde{u}$ the *harmonic* extension of u. It can be checked through direct computation that

$$E_{m+1}(\tilde{u}) = \frac{3}{5} E_m(u)$$

for all functions u defined on Γ_m.

If we successively harmonically extend a function h defined on V_0 to a function h' defined on V^*, h' is called a *harmonic function*. Harmonic functions have the property that their renormalized graph energies remain unchanged after each step. Indeed, if we define the renormalized graph energy $\mathcal{E}_m$ by

$$\mathcal{E}_m(u) = \left(\frac{5}{3}\right)^m E_m(u), \tag{16}$$

harmonic functions have the following property

$$\mathcal{E}_m(h) = \mathcal{E}_0(h) \quad \text{for all integer m.} \tag{17}$$

The converse of 17 holds as well; indeed, every function that satisfies 17 is the repeated harmonic extension of a function defined on V_0 to a function defined on V^*. Hence harmonic functions are defined by equation 17. Harmonic functions can equivalently be defined by the following *mean-value property*: suppose h takes the values a, b and c at the vertices of an arbirtrary cell K_w; the values x, y, z of h at the interior vertices as indicated by Fig. 3 satisfy the following system of equations

$$\begin{aligned} 4x &= b + c + y + z \\ 4y &= a + c + x + z \\ 4z &= a + b + x + y. \end{aligned} \tag{18}$$

This shows that harmonic functions on K are uniquely determined by their value on the boundary V_0. Also, the mean value property guaranties that harmonic functions attain their maximum and minimum at the boundary. We denote the three dimensional vector space of harmonic functions by $\mathcal{H}_0$. In the sequel, we will also consider the space of *harmonic functions of level m* denoted by $S(\mathcal{H}_0, V_m)$ and defined to be the set of functions on V^* such that their restriction to every $m-cell$ is harmonic. In other words, a function u belongs to $S(\mathcal{H}_0, V_m)$ if and only if $u \circ F_w$ is harmonic for every $|w| = m$. Clearly, the set of piecewise harmonic functions of level m has dimension equal to $\#V_m = \frac{1}{2}(3^{m+1} + 3)$.

For every function u defined on K, the sequence $\mathcal{E}_m(u)$ is increasing since

$$\mathcal{E}_{m+1}(u) \geq \mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u),$$

where $\tilde{u}$ is the harmonic extension of u from Γ_m to Γ_{m+1}. Therefore, the sequence $\mathcal{E}_m(u)$ either converges to a real positive number or tends to infinity. This allows us to define a quadratic form $\mathcal{E}$ on K by setting

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u). \tag{19}$$

Similarly, we can define an energy form on K by setting

$$\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u, v). \tag{20}$$

It is easy to see that $\mathcal{E}(u) = 0$ if and only if u is constant on K. A function u belongs to the domain of the energy or has *finite energy* if and only if $\mathcal{E}(u)$ is finite, and we write $u \in \text{Dom}\mathcal{E}$.

If we combine equation 15 with the fact that harmonic extensions are linear (note that equations 18 are linear in a, b and c), i.e $\widetilde{u+v} = \tilde{u} + \tilde{v}$, we obtain

$$\mathcal{E}_{m+1}(\tilde{u}, \tilde{v}) = \frac{1}{4}\left(\mathcal{E}_{m+1}(\tilde{u}+\tilde{v}) - \mathcal{E}_{m+1}(\tilde{u}-\tilde{v})\right) = \frac{1}{4}\left(\mathcal{E}_{m}(\tilde{u}+\tilde{v}) - \mathcal{E}_{m}(\tilde{u}-\tilde{v})\right) = \mathcal{E}_m(u, v). \tag{21}$$

Also, if v' denotes any harmonic extension of v, the following equality holds

$$\mathcal{E}_{m+1}(\tilde{u}, v') = \mathcal{E}_m(u, v). \tag{22}$$

If we set $v'' = v' - \tilde{v}$, then in virtue of equation 21, proving equation 22 becomes equivalent to proving that

$$\mathcal{E}_{m+1}(v'', \tilde{u}) = 0.$$

We have

$$\begin{aligned}\mathcal{E}_{m+1}(v'', \tilde{u}) = \left(\frac{5}{3}\right)^m [\sum_{p\in V_m} \sum_{q\sim_m p} & v''(p)\left(\tilde{u}(p) - \tilde{u}(q)\right) \\ + \sum_{p\in V_{m+1}\backslash V_m} \sum_{q\sim_m p} & v''(p)\left(\tilde{u}(p) - \tilde{u}(q)\right)].\end{aligned}$$

Note that the first sum on the right hand side is equal to zero since $v''|_{V_m} = 0$ (v' and $\tilde{v}$ are both extensions of v from Γ_m to Γ_{m+1}). Furthermore, the second term of the right hand side is also equal to zero since $\tilde{u}$ satisfies the mean value property on $V_{m+1}\backslash V_m$. Therefore, if u and v are real valued functions of K with $u \in \mathcal{H}_0$, we obtain

$$\mathcal{E}(u, v) = \mathcal{E}_0(u, v).$$

Functions in Dom$\mathcal{E}$ have very nice regularity properties. It can be shown (see [5]) that functions of finite energy are continuous on K. Indeed, functions of finite energy are absolutely continuous on K; and since V^* is dense in K with respect to the Euclidean metric (K equipped with the subspace topology) , they are uniquely extandable to continuous functions on K. Also, Dom$\mathcal{E}$ forms a dense subset of $C(K)$ with respect to the uniform convergence metric. In fact, one can actually show that the set of piecewise harmonic functions is dense in $C(K)$ with respect to the uniform metric and dense in Dom$\mathcal{E}$ with respect to the energy metric ($\sqrt{\mathcal{E}(\cdot,\cdot)}$).

We state without proof the following theorem which can be found in [5].

Theorem 3.1. Dom$\mathcal{E}$/*constants equipped with the energy metric* $\sqrt{\mathcal{E}(\cdot, cdot)}$ *forms a Hilbert space.*

Therefore, the quadratic energy $\mathcal{E}$ satisfies the axioms of a Dirichlet form on K: $(\mathrm{Dom}\mathcal{E}/constants, \sqrt{\mathcal{E}(\cdot,\cdot)})$ forms a Hilbert space, and the following property known as *Markov property* holds

$$\mathcal{E}([u]) \leq \mathcal{E}(u) \quad \text{for } u = \min\{1, \max\{u, 0\}\}.$$

Also, $\mathcal{E}$ is *strongly local*, meaning $\mathcal{E}(u, v) = 0$ if v is constant on the support of u. The energy form satisfies the following trivial self-similar identity similar to the one satisfied by integrals and measures:

$$\mathcal{E}(u, v) = \sum_{i=0}^{2} r^{-1}\mathcal{E}(u \circ F_i, v \circ F_i).$$

We are now in a position to define the Laplacian on K. Laplacians on the Sierpinski Gasket will depend on the measure μ used. We will mark the dependence of the Laplacian Δ on the measure μ by denoting it by Δ_μ. When μ is the standard measure, Δ_μ will be called the standard Laplacian.

Definition Let $u \in \mathrm{Dom}\mathcal{E}$ and let f be a continuous function on K. We say that u belongs to the domain of the Laplacian (denoted $\mathrm{Dom}\Delta_\mu$) with $\Delta_\mu u = f$ if

$$\mathcal{E}(u, v) = -\int_K f v d\mu \text{ for all } v \in \mathrm{Dom}\mathcal{E}_0. \tag{23}$$

Here $\mathrm{Dom}\mathcal{E}_0$ denotes the set of all functions of finite energy which are zero on the boundary V_0. To guarantee the uniqueness of the Laplacian f of u, we will assume that the measure μ has full support K. There is an equivalent definition of the Laplacian that does not involve the restriction $v \in \mathrm{Dom}\mathcal{E}_0$. This alternative definition involves an additional term that reflects the behavior of the functions at the boundary:

$$\mathcal{E}(u, v) = -\int_K f v d\mu + \sum_{x \in V_0} v(x)\partial_n u(x) \text{ for all } v \in \mathrm{Dom}\mathcal{E}.$$

Here, for $x \in V_0$, we define $\partial_n u(x)$ by

$$\partial_n u(x) = \lim_{m \to \infty} r^{-m} \sum_{y \sim_m x} (u(x) - u(y)).$$

We call $\partial_n u(x)$ the *normal derivative* of u at x.

The following theorem gives an alternative definition of harmonic functions which is similar to the one found in Euclidean settings. We refer the reader to [5] for a proof.

Theorem 3.2. *Let μ be any regular probability measure on K. A function h is harmonic if and only if $\Delta_\mu h = 0$.*

We now define a class of piecewise harmonic functions, called tent functions, that belong to Dom$\mathcal{E}_0$. Given $x \in V_m$, we define $\psi_x^{(m)}$ as the piecewise harmonic function of level m that satisfies $\psi_x^{(m)}(y) = \delta_{xy}$ for all y in V_m. The definition of the Laplacian given above is weak in the sense that it does not give us a concrete way of computing the pointwise value of the Laplacian. using tent functions, it can be shown that for all $x \in V^* \backslash V_0$

$$\Delta_\mu u(x) = \lim_{m\to\infty} r^{-m} \left(\int_K \psi_x^{(m)} d\mu \right)^{-1} \Delta_m u(x) \tag{24}$$

where $r = 3/5$ and

$$\Delta_m u(x) = \sum_{y \sim_m x} u(y) - u(x).$$

Equation 24 is obtained by substituting $\psi_x^{(m)}$ into the definition of the Laplacian and by observing that

$$\int_K f \psi_x^{(m)} d\mu \approx f(x) \int_K \psi_x^{(m)} d\mu.$$

When μ is the standard measure, the computation

$$\int_K \psi_x^{(m)} d\mu = \frac{2}{3^{m+1}}$$

can help us further simplify equation 24. One then obtains the following pointwise formula for the standard Laplacian

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x). \tag{25}$$

Combining the self-similar property of the measures and integrals, we can establish a self-similar identity for Laplacians when the underlying measure μ satisfies 9. Using the weak formulation of the Laplacian, it is not hard to show that for an arbirtrary word $w \in W^*$ the following identity holds

$$\Delta_\mu(u \circ F_w) = r^{|w|}\mu_w(\Delta_\mu u) \circ F_w. \tag{26}$$

4 Kusuoka Measure and Laplacian

Let $\mathcal{P}$ be a finite set of words that satisfies

$$K = \bigcup_{w\in\mathcal{P}} F_w K$$

where the cells $F_w K$ are disjoint. Using the definition of the graph energy forms $\mathcal{E}_m$, one can easily verify the following additive identity for the energy

$$\mathcal{E}(u) = \sum_{w\in\mathcal{P}} r^{-|w|}\mathcal{E}(u \circ F_w). \tag{27}$$

This identity can be used to define *energy measures* ν_u associated to the functions $u \in \mathrm{Dom}\mathcal{E}$ on K. For a cell $F_w K$, $\nu_u(F_w K)$ is defined in the same manner as $\mathcal{E}(u)$, except that the sum is restricted to the vertices of the graph contained in $F_w K$

$$\nu_u(F_w K) = \lim_{m\to\infty} r^{-m} \sum x \sim_m y, x, y \in F_w K \,(u(x) - u(y))^2\,. \tag{28}$$

An alternative way of defining the measure of a cell $F_w K$ is

$$\nu_u(F_w K) = r^{-|w|}\mathcal{E}(u \circ F_w). \tag{29}$$

The Caratheodory Extension Theorem allows us to extend ν_u to a borel measure on K.

Similarly, given two function $u, v \in \mathrm{Dom}\mathcal{E}$ we can define signed measures $\nu_{u,v}$ on K by assigning to cells $F_w K$ the weights

$$\nu_{u,v}(F_wK) = r^{-|w|}\mathcal{E}(u \circ F_w, v \circ F_wK). \tag{30}$$

The energy measures we have defined are related to the energy form through the following *carreé du champs* formula

$$\int_K f d\nu_{u,v} = \frac{1}{2}\mathcal{E}(fu, v) + \frac{1}{2}\mathcal{E}(u, fv) - \frac{1}{2}\mathcal{E}(f, uv). \tag{31}$$

As we mentioned in the previous section, harmonic functions on K form a three dimensional vector space $\mathcal{H}_\prime$ equipped with the inner product coming from the energy form $\sqrt{\mathcal{E}(\cdot,\cdot)}$. Since constant functions form a one dimensional subspace of $\mathcal{H}_\prime$, the space $\tilde{\mathcal{H}_\prime} := \mathcal{H}_\prime/constants$ represent a two dimensional vector space. If we let $\{h, h'\}$ represent an orthonormal basis for $\tilde{\mathcal{H}_\prime}$, then the measure ν defined by

$$\nu = \nu_h + \nu_{h'} \tag{32}$$

is independent of the choice of the orthonormal basis $\{h, h'\}$. Indeed, let $\{h_1, h_2\}$ be another basis of $\tilde{\mathcal{H}_\prime}$ such that $h_1 = \alpha_1 h + \beta_1 h'$, $h_2 = \alpha_2 h + \beta_2 h'$ and set $\nu' = \nu_{h_1} + \nu_{h_2}$. The matrix $A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$

is a rotation matrix. We have

$$\begin{aligned} \nu'(F_wK) &= r^{-|w|}\left(\mathcal{E}(h_1 \circ F_w) + \mathcal{E}(h_2 \circ F_w)\right) \\ &= r^{-|w|}\left(\alpha_1^2\mathcal{E}(h \circ F_w) + \beta_1^2\mathcal{E}(h' \circ F_w)\right. \\ &+ \alpha_2^2\mathcal{E}(h \circ F_w) + \beta_2^2\mathcal{E}(h' \circ F_w) \\ &+ \alpha_1\beta_1\mathcal{E}(h \circ F_w, h' \circ F_w) + \alpha_2\beta_2\mathcal{E}(h \circ F_w, h' \circ F_w)\left.\right) \end{aligned}$$

for all cells F_wK. Since A is an orthonormal matrix, its coefficients satisfy the following conditions

$$\begin{aligned} \alpha_1^2 + \alpha_2^2 &= 1 \\ \beta_1^2 + \beta_2^2 &= 1 \\ \alpha_1\beta_1 + \alpha_2\beta_2 &= 0. \end{aligned}$$

Hence, we obtain

$$\nu'(F_w K) = \nu_h(F_w K) + \nu_{h'}(F_w K).$$

We call ν the *Kusuoka measure*. We will study its associated domain and Laplacian. The following theorem can be found in [6].

Theorem 4.1. *For any $u \in \mathrm{Dom}\mathcal{E}$, the measure ν_u is singular with respect to the standard measure μ. Furthermore, all energy measures are absolutely continuous with respect to ν.*

We define the Kusuoka Laplacian in accordance with 23. The following lemma gives a pointwise formula for the Kusuoka Laplacian.

Theorem 4.2. *Suppose u belongs to $\mathrm{Dom}\Delta_\nu$. Then for all $x \in V^*\backslash V_0$ the following pointwise formula holds with uniform limit across $V^*\backslash V_0$*

$$\Delta_\nu u(x) = 2 \lim_{m\to\infty} \frac{\Delta_m u(x)}{\Delta_m(h^2 + h'^2)(x)}. \tag{33}$$

Proof. We first recall that for a given measure μ with full support K, the following pointwise estimate holds uniformly for points in $V^*\backslash V_0$:

$$\Delta_\mu u(x) = \lim_{m\to\infty} r^{-m} \left(\int_K \psi_x^{(m)} d\mu \right)^{-1} \Delta_m u(x).$$

In our case, to refine this pointwise formula we have to compute the integral

$$\int_K \psi_x^{(m)} d\mu$$

with μ equal to the Kusuoka measure. Using the *carré du champs* formula, we obtain

$$\begin{aligned} \int_K \psi_x^{(m)} d\nu &= \mathcal{E}(\psi_x^{(m)} h, h) + \mathcal{E}(\psi_x^{(m)} h', h') \\ &\quad - \frac{1}{2}\mathcal{E}(\psi_x^{(m)}, h^2) - \frac{1}{2}\mathcal{E}(\psi_x^{(m)}, h'^2). \end{aligned}$$

Equation 22 combined with the fact that $\psi_x^{(m)} h, \psi_x^{(m)} h' \in \mathrm{Dom}_0\mathcal{E}$ gives

$$\begin{aligned} \mathcal{E}(\psi_x^{(m)} h, h) &= \mathcal{E}_0(\psi_x^{(m)} h, h) = 0 \\ \mathcal{E}(\psi_x^{(m)} h', h') &= \mathcal{E}_0(\psi_x^{(m)} h', h') = 0. \end{aligned}$$

Our equation thus simplifies to

$$\int_K \psi_x^{(m)} d\nu = -\frac{1}{2}\mathcal{E}_m(\psi_x^{(m)}, h^2 + h'^2)$$
$$= r^{-m}\Delta_m(h^2 + h'^2).$$

Therefore, we get

$$\Delta_\nu u(x) = \lim_{m\to\infty} r^{-m}\left(\int_K \psi_x^{(m)} d\nu\right)^{-1} \Delta_m u(x) = 2\lim_{m\to\infty} \frac{\Delta_m u(x)}{\Delta_m(h^2 + h'^2)(x)}.$$

We now turn our attention to the domains of the Kusuoka Laplacian and the standard Laplacian. We know that their intersection contains the space $\mathcal{H}_0$; our next theorem shows that the intersection is actually equal to $\mathcal{H}_0$.

Theorem 4.3. $\mathrm{Dom}\Delta_\nu \cap \mathrm{Dom}\Delta = \mathcal{H}_0$

Proof. Suppose u belongs to $\mathrm{Dom}\Delta_\nu \cap \mathrm{Dom}\Delta$ and $u \notin \mathcal{H}_0$. Let $\{h, h'\}$ be an orthonormal basis for $\tilde{\mathcal{H}}_{\prime}$. We claim that there exists a point $x \in V^*\backslash V_0$ such that $\Delta_\nu u(x) \neq 0$ and $\partial_\nu h(x) \neq 0$. Indeed, since u is not harmonic and belongs to the domain of the Kusuoka Laplacian, there exists a cell $F_w K$ such that $\Delta_\nu u(y) \neq 0$ for all $y \in F_w K$. Since h is not constant, its restriction to all cells are not constant. Hence, on every cell, the partial derivative of h is non-zero on at least one vertex (otherwise h would be constant on that cell). Therefore, if we look at the cell $F_w K$ where $\Delta_\nu u \neq 0$, there is at least one $x \in F_w K \cap V^*\backslash V_0$ such that $\partial_\nu h(x) \neq 0$. We have

$$\Delta_\nu u(x) = 2\lim_{m\to\infty} \frac{\Delta_m u(x)}{\Delta_m(h^2 + h'^2)(x)}$$
$$= 2\lim_{m\to\infty} \frac{5^m\Delta_m u(x)}{5^m\Delta_m(h^2 + h'^2)(x)} \neq 0.$$

The limit of the numerator exists

$$\lim_{m\to\infty} 5^m\Delta_m u(x) < \infty$$

since $u \in \mathrm{Dom}\Delta$.

The limit of the denominator must also exist and be different from infinity:

$$\lim_{m\to\infty} 5^m\Delta_m(h^2 + h'^2)(x) \neq \infty.$$

However, we have

$$\lim_{m\to\infty} 5^m \Delta_m h^2(x) = \lim_{m\to\infty} 5^m \left(\sum_{y\sim_m x} (h(y)-h(x))^2 + 2h(x) \sum_{y\sim_m x} (h(y)-h(x)) \right)$$
$$= \lim_{m\to\infty} 5^m \sum_{y\sim_m x} (h(y)-h(x))^2$$

where we have used the fact that $\Delta_m h(x) = 0$ since h is harmonic.
Now the assumption $\partial_\nu h(x) \neq 0$ implies that for large $n's$, there is a sequence of vertices $x_n \in \Gamma_n$ such that

$$\left(\frac{5}{3}\right)^m |h(x) - h(x_n)| \geq c$$

for some positive constant c. We have

$$\lim_{m\to\infty} 5^m \sum_{y\sim_m x} (h(y)-h(x))^2 \geq \lim_{m\to\infty} 5^m |h(x_n) - h(x)|^2$$
$$\geq c((3/5)^2 5)^n \to \infty$$

since $9/5 > 1$.
Therefore, $5^m \Delta_m (h^2 + h'^2)(x)$ diverges to ∞; note that $\Delta_m h'^2 > 0$ for all m. This shows that $\Delta_\nu u(x) = 0$, which contradicts our assumptions. We conclude that $\mathrm{Dom}\Delta \cap \mathrm{Dom}\Delta_\nu \subset \mathcal{H}_0$.

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