WPI Department of Mathematical Sciences 503 GCE
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Name:_______________________________

Exercise 1:

1. Show that if $f_n \to f$ uniformly on $\mathbb{R}$, each f_n is continuous, and $x_n \to x$, then $f_n(x_n) \to f(x)$.
2. Give an example of $\{f_n\}, f, \{x_n\}$ satisfying the conditions of 1., except $f_n \to f$ only pointwise, for which it is not true that $f_n(x_n) \to f(x)$.

Exercise 2:

Let $(X, \mathcal{A})$ be a measure space and α, β, two finite measures on $\mathcal{A}$. Let $\mathcal{E}$ be a subset of $\mathcal{A}$ such that the smallest σ-algebra in $\mathcal{P}(X)$ containing $\mathcal{E}$ is $\mathcal{A}$. Assume that
(i). $\alpha \leq \beta$,
(ii). for all E in $\mathcal{E}$, $\alpha(E) = \beta(E)$,
(iii). $\alpha(X) = \beta(X)$.
Set $\mathcal{T} = \{T \in \mathcal{A} : \alpha(T) = \beta(T)\}$.

1. Let S, T be in $\mathcal{T}$ such that $S \cap T = \emptyset$. Show that $S \cup T$ is in $\mathcal{T}$.
2. Let T be in $\mathcal{T}$ such that there are A_1, A_2 in $\mathcal{A}$ satisfying $T = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$. Show that A_1 and A_2 are in $\mathcal{T}$.
3. Let S, T be in $\mathcal{T}$. Show that $T \setminus S$, $T \cap S$, and $T \cup S$ are in $\mathcal{T}$.
4. Show that $\mathcal{T}$ is a σ- algebra and infer that $\alpha = \beta$.

Exercise 3:

Let the functions $f_n \in L^p([0,1])$ for some $1 < p < \infty$. Assume that $\sup_n \|f_n\|_{L^p([0,1])} < \infty$.

1. Show that the sequence is equiintegrable, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for $A \subset [0,1]$,

$$m(A) < \delta \Rightarrow \int_A |f_n| < \varepsilon$$

$\forall n \in \mathbb{N}$. (Hint: write $\int_A |f_n| = \int |f_n| \chi_A$, and use Hölder's inequality).

2. Assume in addition that $f_n \to f$ in measure on $[0,1]$ for some function f, i.e., $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} m(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Show that $f_n \to f$ in $L^1([0,1])$, i.e., $\lim_{n \to \infty} \|f_n - f\|_{L^1([0,1])} = 0$.