Worcester Polytechnic Institute
Department of Mathematical Sciences
January 9, 2023 NAME: ______________________

General Comprehensive Exam
Linear Algebra

Problem 1 Let $K^{n\times n}$ be the vector space of square n by n matrices with entries in the field K. Let $e_1, ..., e_n$ be the natural basis of K^n and E_{ij} be the matrix $e_i e_j^T$, $1 \le i, j \le n$.

(a) Show how the product $E_{ij}E_{kl}$ can be simplified where $1 \le i, j, k, l \le n$.

(b) Let f be a linear map from $K^{n\times n}$ to K such that for all A and B in $K^{n\times n}$, $f(AB) = f(BA)$. Show that f is a multiple of the trace.

Problem 2 Let A be an $n \times n$ Hermitian matrix with complex entries.

(a) Prove that every eigenvalue of A is real.

(b) Prove: *if* $\mathbf{u}$ *and* $\mathbf{v}$ *are eigenvectors for* A *belonging to distinct eigenvalues, then* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal.*

Problem 3 Let A and B be symmetric $n \times n$ matrices with real entries.

(a) Prove: *if* A *and* B *commute, they are simultaneously diagonalizable.* That is, if $AB = BA$, then there exists a basis for $\mathbb{R}^n$ where each vector in the basis is an eigenvector for both A and B.

(b) Illustrate, via a small counterexample, that $AB = BA$ is a necessary condition.

Problem 4 Let A be an $n \times n$ real matrix and $A^\top$ its transpose. Show that $A^\top A$ and $A^\top$ have the same range.

Problem 5 Let n be a positive integer, and let $A = (a_{ij})_{i,j=1}^n$ be the $n \times n$ matrix with $a_{ii} = 2, a_{ii\pm 1} = -1$, and $a_{ij} = 0$ otherwise; that is,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Prove that every eigenvalue of A is a positive real number.

Problem 6 Consider the vector space $\mathcal{C}_P[0, 2\pi] = \{f \mid f : [0, 2\pi] \to \mathbb{R} \text{ pcf}\}$ of all piecewise continuous real-valued functions defined on the interval $[0, 2\pi]$. The addition in this vector space is usual addition of functions and scalar multiplication is as usual also: $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for $x \in [0, 2\pi]$ and cf is the function $(cf)(x) = c(f(x))$ for $x \in [0, 2\pi]$.

The inner product we consider on this space is

$$\langle f(x), g(x)\rangle := \int_0^{2\pi} f(x)g(x)\,dx.$$

(a) Let W be the subspace of $\mathcal{C}_P[0, 2\pi]$ spanned by the eight vectors

$$\cos(x), \sin(x), \quad \cos(2x), \sin(2x), \quad \ldots, \cos(4x), \sin(4x)\ .$$

Find an orthonormal basis for this subspace. *[HINT: You don't have to do too much to adjust the above basis. Use standard formulas for integrals.]*

(b) Consider the function

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x < \pi; \\ 1 & \text{if } \pi \le x \le 2\pi. \end{cases}$$

Find the projection of $f(x)$ onto subspace W.