Problem 1 Let $K^{n \times n}$ be the vector space of square $n$ by $n$ matrices with entries in the field $K$. Let $e_1, ..., e_n$ be the natural basis of $K^n$ and $E_{ij}$ be the matrix $e_ie_j^T$, $1 \leq i, j \leq n$.

(a) Show how the product $E_{ij}E_{kl}$ can be simplified where $1 \leq i, j, k, l \leq n$.

(b) Let $f$ be a linear map from $K^{n \times n}$ to $K$ such that for all $A$ and $B$ in $K^{n \times n}$, $f(AB) = f(BA)$. Show that $f$ is a multiple of the trace.

Problem 2 Let $A$ be an $n \times n$ Hermitian matrix with complex entries.

(a) Prove that every eigenvalue of $A$ is real.

(b) Prove: if $u$ and $v$ are eigenvectors for $A$ belonging to distinct eigenvalues, then $u$ and $v$ are orthogonal.

Problem 3 Let $A$ and $B$ be symmetric $n \times n$ matrices with real entries.

(a) Prove: if $A$ and $B$ commute, they are simultaneously diagonalizable. That is, if $AB = BA$, then there exists a basis for $\mathbb{R}^n$ where each vector in the basis is an eigenvector for both $A$ and $B$.

(b) Illustrate, via a small counterexample, that $AB = BA$ is a necessary condition.

Problem 4 Let $A$ be an $n \times n$ real matrix and $A^\top$ its transpose. Show that $A^\top A$ and $A^\top$ have the same range.

Problem 5 Let $n$ be a positive integer, and let $A = (a_{ij})_{i,j=1}^n$ be the $n \times n$ matrix with $a_{ii} = 2$, $a_{ii \pm 1} = -1$, and $a_{ij} = 0$ otherwise; that is,

$$A = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \\
\end{pmatrix}.$$

Prove that every eigenvalue of $A$ is a positive real number.
Problem 6 Consider the vector space $C_P[0,2\pi] = \{ f \mid f : [0,2\pi] \to \mathbb{R} \text{ pcf} \}$ of all piecewise continuous real-valued functions defined on the interval $[0,2\pi]$. The addition in this vector space is usual addition of functions and scalar multiplication is as usual also: $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for $x \in [0,2\pi]$ and $cf$ is the function $(cf)(x) = c(f(x))$ for $x \in [0,2\pi]$.

The inner product we consider on this space is

$$\langle f(x), g(x) \rangle := \int_0^{2\pi} f(x)g(x) \, dx.$$ 

(a) Let $W$ be the subspace of $C_P[0,2\pi]$ spanned by the eight vectors

$$\cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(4x), \sin(4x).$$

Find an orthonormal basis for this subspace. [HINT: You don’t have to do too much to adjust the above basis. Use standard formulas for integrals.]

(b) Consider the function

$$f(x) = \begin{cases} 
-1 & \text{if } 0 \leq x < \pi; \\
1 & \text{if } \pi \leq x \leq 2\pi.
\end{cases}$$

Find the projection of $f(x)$ onto subspace $W$. 

2