

GCE - Linear Algebra  
August 2023  
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Exercise 1

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{C}$  and  $A$  in  $\mathcal{L}(V)$  a linear operator.

(i). Assume that  $A = \lambda I + N$  where  $\lambda \in \mathbb{C}$ ,  $I$  is the identity operator, and  $N$  is nilpotent. Show that  $V$  is the direct sum of  $\text{Ker } A^n$  and  $\text{Im } A^n$ .

(ii). Is  $V$  still the direct sum of  $\text{Ker } A^n$  and  $\text{Im } A^n$  for any  $A$  in  $\mathcal{L}(V)$  ?

Exercise 2

Let  $K$  be  $\mathbb{R}$  or  $\mathbb{C}$  and  $A$  an  $n$  by  $n$  matrix with entries in  $K$ . Let  $A = (a_{ij})_{1 \leq i, j \leq n} \in K^{n \times n}$

and  $\|A\|_\infty = \sup_{x \in K^n, \|x\|_\infty=1} \|Ax\|_\infty$ . Show that  $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$ .

Exercise 3

Prove that an  $m$  by  $n$  matrix  $A$  with real entries has rank less or equal than  $r$  if and only if  $A$  can be expressed as a sum of  $r$  rank one matrices.

Exercise 4

Let  $A$  be an  $n$  by  $n$  real matrix and  $A^t$  its transpose. Show that  $A^t A$  and  $A^t$  have same range.