GCE - Linear Algebra
May 2023

Exercise 1

Let V be a finite-dimensional vector space and W a subspace of V. Let T and S be linear transformations on V.

(i). Show that $S^{-1}W = \{x \in V : Sx \in W\}$ is a subspace of V.

(ii). Show that

$$\dim S^{-1}W \leq \dim \text{Ker } S + \dim W$$

(iii). Show that the composition $T \circ S$ satisfies

$$\dim \text{Ker } S \leq \dim \text{Ker } T \circ S \leq \dim \text{Ker } T + \dim \text{Ker } S.$$

Exercise 2

Let V be a finite-dimensional space over $\mathbb{R}$ with $\dim V = n$. Let $<,>$ be a positive definite scalar product on $V \times V$ and A a symmetric linear map from V to V. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of A.

(i). Show that

$$\lambda_n = \sup_{x \in V, \|x\|=1} < Ax, x > .$$

(ii). Let H be a subspace of V with dimension $n-1$. Let w_1, w_2 be two independent vectors in V. Show that there is some x in $\text{span}\{w_1, w_2\} \cap H$ such that $\|x\| = 1$.

(iii). Let S be the set of all subspaces of V with dimension $n-1$. Show that

$$\lambda_{n-1} = \inf_{H \in S} \sup_{x \in H, \|x\|=1} < Ax, x > .$$

Exercise 3

Let A in $\mathbb{C}^{n\times p}$ and B in $\mathbb{C}^{p\times n}$ be two matrices.

(i). Let λ be an eigenvalue of $I + AB$ such that $\lambda \neq 1$. Show that λ is an eigenvalue of $I + BA$.

(ii). Show that $\det(I + AB) = \det(I + BA)$.

Exercise 4

Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a real matrix such that $a_{ij} > 0$ for all $1 \leq i, j \leq n$. Recall the definition of the spectral radius

$$\rho(A) = \max\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ is an eigenvalue of } A\},$$

and the property $\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$ where $\| \, \|$ is a multiplicative norm.

Let $v = (v_1, ..., v_n)$, $w = (w_1, ..., w_n)$ be two vectors in $\mathbb{R}^n$. We say that $v > w$ if $v_i > w_i$ for all $i = 1, ..., n$. $v \geq w$, $v < w$, $v \leq w$ are defined likewise.

(i). Let v be a vector in $\mathbb{R}^n$ such that $v \geq 0$ and $v \neq 0$. Show that $Av > 0$.
(ii). Let $v = (v_1, ..., v_n)$ be a vector in $\mathbb{C}^n$. We define the vector $|v| = (|v_1|, ..., |v_n|)$. Show that $|Av| \leq A|v|$.
(iii). Show that $\rho(A) > 0$.
(iv). Set $B = (\rho(A))^{-1}A$. Let x in $\mathbb{C}^n$ be such that $Bx = \lambda x$, $x \neq 0$ and $|\lambda| = 1$. Show that $|x|$ is an eigenvector of B and conclude. **Hint:** Set $y = B|x| - |x|$.