Real Analysis GCE Name:__________________________

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Exercise 1:

Suppose that $f_n, f : \mathbb{R} \to \mathbb{R}$, and $f_n \to f$ uniformly. Show that

$$\sup_{x\in\mathbb{R}} f_n(x) \to \sup_{x\in\mathbb{R}} f(x).$$

Exercise 2:

Suppose $f_n, f \in L^1(\mathbb{R})$ with $f_n \geq 0$ and $f_n \to f$ a.e. Suppose also that

$$m(\{x \in \mathbb{R} : f_n(x) > f(x) + \alpha\}) \to 0$$

for all $\alpha > 0$. Prove or give a counterexample that $\int f_n \to \int f$ must follow.

Exercise 3:

Let E be a subset (not necessarily Lebesgue measurable) of $[0, 1]$.

1. Suppose the set E has positive outer measure, i.e. $m^*(E) > 0$. Prove that for any $0 < \alpha < 1$, there exists an open interval $I \subset [0, 1]$ such that
$$m^*(E \cap I) \geq \alpha\, m(I).$$
2. Suppose E is Lebesgue measurable, and there exists $0 < \alpha_0 < 1$ such that for any nonempty open interval $I \subset [0, 1]$, we have
$$m(E \cap I) > \alpha_0\, m(I).$$
Prove that $m(E) = 1$. (You may use the conclusion of part 1 even if you cannot prove it.)