Worcester Polytechnic Institute
Department of Mathematical Sciences

Jan 2022

General Comprehensive Examination

Linear Algebra

Print Name: ____________________________ Sign: ____________________________

No documents, no calculators allowed.
Choose any five problems among the six problems listed below.
Unless otherwise specified, all matrices are assumed to have complex entries.

Problem 1: Let A be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find all the eigenvalues and eigenspaces of A and compute $\det(A)$.

Problem 2: Assume that $A \in \mathbb{C}^{n \times n}$ is a normal matrix. Prove that if (λ, v) is an eigenpair for A then $(\bar{\lambda}, v)$ is eigenpair for A^* (the adjoint of A).

Problem 3: Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^k x = 0$ and $T^{k-1}x \neq 0$ for some integer $k > 0$. Show that the set $\{x, Tx, \ldots, T^{k-1}x\}$ is linearly independent.

Problem 4: Let A be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix Q and a unique positive definite symmetric matrix S such that $A = QS$.

Problem 5: Let A be an $n \times n$ Hermitian matrix with largest eigenvalue λ_1. Let B be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of A. If μ_1 is the largest eigenvalue of B, prove that $\mu_1 \leq \lambda_1$.

Problem 6: Let S, T be two normal operators on the finite-dimensional complex inner product space V such that $ST = TS$. Prove that there is a basis for V consisting of vectors that are eigenvectors of both S and T.