Problem 1: Let $A$ be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find all the eigenvalues and eigenspaces of $A$ and compute $\det(A)$.

Problem 2: Assume that $A \in \mathbb{C}^{n \times n}$ is a normal matrix. Prove that if $(\lambda, v)$ is an eigenpair for $A$ then $(\bar{\lambda}, v)$ is eigenpair for $A^*$ (the adjoint of $A$).

Problem 3: Let $T$ be a linear transformation of a vector space $V$ into itself. Suppose $x \in V$ is such that $T^k x = 0$ and $T^{k-1} x \neq 0$ for some integer $k > 0$. Show that the set \( \{x, Tx, \ldots, T^{k-1}x\} \) is linearly independent.

Problem 4: Let $A$ be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix $Q$ and a unique positive definite symmetric matrix $S$ such that $A = QS$.

Problem 5: Let $A$ be an $n \times n$ Hermitian matrix with largest eigenvalue $\lambda_1$. Let $B$ be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of $A$. If $\mu_1$ is the largest eigenvalue of $B$, prove that $\mu_1 \leq \lambda_1$.

Problem 6: Let $S, T$ be two normal operators on the finite-dimensional complex inner product space $V$ such that $ST = TS$. Prove that there is a basis for $V$ consisting of vectors that are eigenvectors of both $S$ and $T$. 