WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics I
January, 2022

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) (a) Let A, B and C be three independent events. Prove that $A \cup B$ and C are independent to each other.
   (b) Consider randomly shuffle all the letters from "ARRANGE". How many distinct 7-letter words can you create? What is the chance that you will get a 7-letter word starting with "A"?

2. (20 points) Consider a sequence of continuous random variables X_n, $n = 1, 2, ...$ Event A: X_n converges in probability to a constant a. Event B: X_n converges in distribution to a.
   (a) Express events A and B in proper math notations.
   (b) Prove that A and B are equivalent, i.e., A if and only if B.

3. (20 points) Suppose that $X \sim Exp(1)$ (i.e., the density function of X is $f(x) = \exp(-x)$ for $x \geq 0$) and S is a discrete random variable taking values -1 and 1 with equal probability (i.e., $P(S = 1) = P(S = -1) = 0.5$), and that S is independent of X. Let $Y = SX$.
   (a) Find the density function $h(y)$ of Y.
   (b) Suppose further that $Z|Y \sim U(0, 2h(Y))$; that is, conditional on Y, Z is uniformly distributed on $[0, 2h(Y)]$. Find $P(Z < \phi(Y))$, the probability of $\{Z < \phi(Y)\}$, where $\phi(\cdot)$ is the density function of the standard normal distribution.

4. (20 points) Say the vector (X, Y) is distributed as bivariate normal, with $\mathbb{E}X = \mathbb{E}Y = 0$, $Var(X) = Var(Y) = 1$ and $cor(X, Y) = \rho \in (-1, 1)$. That is,

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp \left\{ -\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)} \right\}$$

   (a) Find the conditional pdf $f_{Y|X}(y|x)$, and show that $\mathbb{E}[Y|X = x] = \rho x$.
   (b) *Regression to the mean.* Say that students' midterm and final exam scores in a class are distributed as above, with $0 < \rho < 1$. For the subset of students who scored above average on the midterm, would you expect their final scores, on average, to be higher or lower than their midterm scores? Would you expect their final scores to be above or below average? Why?

5. (20 points) Let $0 < t < T < \infty$ and let A denote the region in the x-y plane given by $A = \{(x, y) : t < x^2 + y^2 < T\}$.

(a) Let (X, Y) denote a point randomly chosen from A. Consider the jointly distributed random variables (R, Θ), where $R \in [t, T]$ is the Euclidean distance of (X, Y) from $(0, 0)$ and $\Theta \in [0, \pi]$ is its angle with the x axis. Find the joint pdf of (R, Θ).

(b) Using the results from (a), explain how, given independent $U(0, 1)$ random variates U and V, you can obtain a random vector (X, Y) uniformly distributed on A.

6. (20 points) Let $X, Y \stackrel{ind}{\sim} \text{Uniform}(0, 1)$ and let $W = XY$. Find the probability density function (pdf) of W. Use this pdf to find $\text{E}(W)$. How do you know that you are correct?