WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics I
August, 2023

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Counting and probability. [You can just give the expressions without calculating the final values.]
   (a) How many distinct words can we get by arranging the letters in the word "Hammedd"?
   (b) An integers is randomly drawn from 7500 to 9999 (including them), what is the chance that it has *all distinct* digits?

2. (20 points) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from uniform$(\theta, 2\theta)$, θ is real.

   a) Find a limiting distribution of $\sqrt{n}(\log \bar{X} - c)$ for an appropriate constant c.

   b) Let $T = aX$ be an estimator of θ where a is a constant. Find the value a such that minimizes the mean square error (MSE). Show that your answer is the minimizer.

3. (20 points) Let $X_1, X_2 \overset{iid}{\sim} Exp(1)$ with the pdf $f_X(x) = e^{-x}$. Show that if $Y = X_1 - X_2$ then Y has the standard Laplace distribution, i.e., $f_Y(y) = \frac{1}{2}e^{-|y|}$.

4. (20 points) Suppose that X follows Pareto(θ) distribution, i.e., the density function of X is
$$f(x) = \frac{\theta}{x^{\theta+1}}, \quad x > 1, \theta > 1.$$
   Let X' be an independent, identically distributed copy of X. Consider the Gini coefficient γ of X, which is defined by
$$\gamma = \frac{1}{2}\frac{\mathrm{E}(|X - X'|)}{\mu}$$
   with $\mu = \mathrm{E}(X)$.

   1. What's the range of γ?
   2. Show that the value of γ is given by
$$\gamma = \frac{1}{2\theta - 1}.$$

5. (20 points) Let constants p and q be constants satisfying $p > 1$ and $1/p + 1/q = 1$. For any two random variables X and Y, prove that

$$E|XY| \leq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}.$$

[You can use the inequality: $x^t y^{1-t} \leq tx + (1-t)y$, where x and y are any nonnegative numbers and $t \in (0,1)$.]

6. (20 points) Let $Y = \log(X)$, where X is a positive continuous random variable.

   a. Assume that $Y \sim \text{Normal}(0,1)$. Find $E(X^t), t = 1, 2, \ldots$. Does the moment generating function of X exist? (Simply answer Yes or No.) Give an obvious reason.

   b. Now assume that $Y \mid t^2 \sim \text{Normal}(0, t^2)$ and $t^{-2} \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$. Give a clear proof to show that the MGF of X does not exist.