WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics I
January, 2023

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Suppose that $X \sim \Gamma(a, b)$, i.e., the density function of X is $f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ for $x > 0$.
   (a) Find $\mathrm{E}(X^k)$, $k = 1, 2, 3, \ldots$.
   (b) Let X_i, $i = 1, 2, \ldots, n$, be i.i.d. $\Gamma(a, b)$. Find the distribution of $Z = \frac{X_1}{\sum_{i=1}^n X_i}$.

2. (20 points) Suppose that U_1 and U_2 are uniformly distributed over $[0, 1]$ and are independent of each other; that is $U_1, U_2 \overset{ind}{\sim} U(0, 1)$.
   (a) Find the distribution of $X_1 = \sqrt{-2 \log U_1} \cos(2\pi U_2)$.
   (b) Find the distribution of $X_2 = \sqrt{-2 \log U_1} \sin(2\pi U_2)$.
   (c) Find the conditional distribution of $X_1 | X_2$.

3. (20 points) The joint probability mass function of $Y_1, \ldots, Y_n$ is

$$f(y) = \frac{B(s + \alpha, n - s + \beta)}{B(\alpha, \beta)}, y_i = 0, 1, i = 1, \ldots, n,$$

   where $B(\cdot, \cdot)$ is the beta function and $s = \sum_{i=1}^n y_i$. Find the random variable, Z, such that $Y_i \mid Z$ are independent and identically distributed. Hence, deduce $\mathrm{Cor}(Y_i, Y_j), i, j = 1, \ldots, n$.

4. (20 points) Suppose that $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$ are independent normal random variables.
   (a) Characterize the collection of all real constants c_1, c_2, c_3, c_4 such that the random variables

$$L_1 = c_1 X + c_2 Y \text{ and } L_2 = c_3 X + c_4 Y$$

   are independent.
   (b) Are the random variables

$$V = \frac{(X + Y)}{\sqrt{2}\sigma} \text{ and } W = \frac{(X - Y)}{\sqrt{2}\sigma}$$

   independent?
   (c) Under the case $\mu_1 = \mu_2$, determine the marginal distributions of V and W.

5. (20 points) *The Truncated Standard Normal Distribution.* Let $Z \sim \mathcal{N}(0, 1)$. For a real number b, let $X_b = Z | Z < b$, that is, Z constrained to be less than b.

(a) Find the pdf of X_b.

(b) Find $\mathbb{E}[X_b]$.

(c) What is $\mathbb{E}[X_0]$, i.e. with $b = 2$?

Your answers for the first two parts may include functions $\phi(\cdot)$ and $\Phi(\cdot)$, the standard normal pdf and cdf, respectively.

6. (20 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from a Poisson distribution with mean θ. We are interested in estimating $g(\theta) = P(X_1 = 0) = e^{-\theta}$. Consider the following two estimators:

$$T_{n,1} = e^{-\bar{X}_n} \qquad \text{and} \qquad T_{n,2} = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 0),$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and I is an indicator function.

a) Find the asymptotic distribution of $T_{n,1}$.

b) Find the asymptotic distribution of $T_{n,2}$.

c) Which estimator is more efficient in estimating $g(\theta)$ when a large sample size is available? Show your argument clearly.