1. (20 points) Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = P(X = x) = \frac{1}{x^\nu \zeta(\nu)},$$

where $\nu > 1$ and $x = 1, 2, 3, \ldots$. Here the zeta function

$$\zeta(\nu) = \sum_{x=1}^{\infty} \frac{1}{x^\nu},$$

for $\nu > 1$.

a) Find a minimal sufficient statistic for $\nu$.

b) Is the statistic found in a) complete? (prove or disprove)

c) Give an example of a sufficient statistic that is strictly not minimal.

2. (20 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$ is a parameter. Let

$$T = \max_{1 \leq i \leq n} X_i$$

and let

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

be the sample mean of $X_i$.

Find $E(\bar{X}|T = t)$ without trying to find the conditional distribution of $X$ given $T = t$.

3. (20 points) Suppose that $X$ follows the Pareto($\theta$) distribution, i.e., the density function of $X$ is

$$f(x) = \frac{\theta}{x^{\theta+1}}, \quad x > 1, \theta > 1.$$

Let $X'$ be an independent, identically distributed copy of $X$. Consider the Gini coefficient $\gamma$ of $X$, which is defined by

$$\gamma = \frac{1}{2} \frac{E(|X - X'|)}{\mu}$$

with $\mu = E(X)$.
(a) Let $X_1, \ldots, X_n$ be an iid sample from the Pareto($\theta$) distribution. Find the maximum likelihood estimator $\hat{\gamma}$ of $\gamma$ based on $X_1, \ldots, X_n$. [Note that $\gamma = \frac{1}{2\theta - 1}$.

(b) Show that $\hat{\gamma}$ is an asymptotically normal estimator of $\gamma$, and find its asymptotic variance.

4. (20 points) Consider testing $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ using one observation $X$ from the density:

$$f_X(x; \theta) = \begin{cases} \frac{\theta e^{\theta x}}{2 \sinh(\theta)} & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\sinh(\theta) = \frac{e^\theta - e^{-\theta}}{2}$ is the hyperbolic sine function.

(a) Find the UMP level-$\alpha$ test for $0 < \alpha < 1$.

(b) For a given $X = x$ and $\alpha$, what inequality would you need to solve in order to invert the test from part (a) to make a $1 - \alpha$ upper confidence bound for $\theta$?

5. (20 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim N(i\theta, 1)$ for $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimator of $\theta$.

(b) Find the variance of MLE from part (a).

(c) Compare this variance from part (b) with the Cramer-Rao lower bound for unbiased estimation of $\theta$.

6. (20 points) Consider independent estimators of $\hat{\theta}_i$, $i = 1, \ldots, \ell$, with mean $\theta$ and variance $\sigma^2/n_i$ (known). Consider the combined estimators of $\theta$ of the form, $\hat{\theta} = \sum_{i=1}^{\ell} a_i \hat{\theta}_i$, where $a_i \geq 0$ are unknown constants.

(a) Obtain the minimum variance unbiased estimator of $\theta$. Describe the estimator and give its standard error. What is a good thing about this estimator?

(b) Find the least squares estimator (LSE) of $\theta$. Is it different from the estimator in (a)?

(c) Discuss how the Central Limit Theorem allows you to approximate the distribution of $\hat{\theta}$?