WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics II
August, 2023

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = P(X = x) = \frac{1}{x^\nu \zeta(\nu)},$$

where $\nu > 1$ and $x = 1, 2, 3, \cdots$. Here the zeta function

$$\zeta(\nu) = \sum_{x=1}^{\infty} \frac{1}{x^\nu},$$

for $\nu > 1$.

a) Find a minimal sufficient statistic for ν.

b) Is the statistic found in a) complete? (prove or disprove)

c) Give an example of a sufficient statistic that is strictly not minimal.

2. (20 points) Let $X_1, X_2, \cdots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$ is a parameter. Let

$$T = \max_{1 \le i \le n} X_i$$

and let

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

be the sample mean of X_i

Find $\mathrm{E}(\bar{X} | T = t)$ **without** trying to find the conditional distribution of X given $T = t$.

3. (20 points) Suppose that X follows the Pareto(θ) distribution, i.e., the density function of X is

$$f(x) = \frac{\theta}{x^{\theta+1}}, \quad x > 1, \theta > 1.$$

Let X' be an independent, identically distributed copy of X. Consider the Gini coefficient γ of X, which is defined by

$$\gamma = \frac{1}{2} \frac{\mathrm{E}(|X - X'|)}{\mu}$$

with $\mu = \mathrm{E}(X)$.

(a) Let $X_1, \ldots, X_n$ be an iid sample from the Pareto(θ) distribution. Find the maximum likelihood estimator $\hat{\gamma}$ of γ based on $X_1, \ldots, X_n$. [Note that $\gamma = \frac{1}{2\theta - 1}$.]

(b) Show that $\hat{\gamma}$ is an asymptotically normal estimator of γ, and find its asymptotic variance.

4. (20 points) Consider testing $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ using one observation X from the desnity:

$$f_X(x;\theta) = \begin{cases} \frac{\theta e^{\theta x}}{2sinh(\theta)} & |x| \leq 1 \\ 0 & otherwise \end{cases}$$

where $sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$ is the hyperbolic sine function.

(a) Find the UMP level-α test for $0 < \alpha < 1$.

(b) For a given $X = x$ and α, what inequality would you need to solve in order to invert the test from part (a) to make a $1 - \alpha$ upper confidence bound for θ?

5. (20 points) Suppose that $X_1, X_2, \cdots, X_n$ are independent random variables with $X_i \sim N(i\theta, 1)$ for $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimator of θ.

(b) Find the variance of MLE from part (a).

(c) Compare this variance from part (b) with the Cramer-Rao lower bound for unbiased estimation of θ.

6. (20 points) Consider independent estimators of $\hat{\theta}_i$, $i = 1, \ldots, \ell$, with mean θ and variance σ^2/n_i (known). Consider the combined estimators of θ of the form, $\hat{\theta} = \sum_{i=1}^{\ell} a_i \hat{\theta}_i$, where $a_i \geq 0$ are unknown constants.

(a) Obtain the minimum variance unbiased estimator of θ. Describe the estimator and give its standard error. What is a good thing about this estimator?

(b) Find the least squares estimator (LSE) of θ. Is it different from the estimator in (a)?

(c) Discuss how the Central Limit Theorem allows you to approximate the distribution of $\hat{\theta}$?