WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics I
May, 2024

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Say $X \sim \mathcal{N}(0,1)$ and $Y \sim exp(1)$ and that X and Y are independent. Let $Z = \frac{X}{Y+1}$. Find $E[Z]$ and $Var(Z)$. You may express your answer in terms of the "exponential integral" $Ei(r) = -\int_{-r}^{\infty} e^{-t}/t dt$.

2. (20 points) Suppose that (X,Y) has a joint probability density function $f_{XY}(x,y) = 8xy\mathbb{I}(0 \leq x \leq y \leq 1)$.
   (a) Let $g(x) = E(Y|X=x)$. Show that for any function $h : (0,1) \to \mathbb{R}$, we have $E[(Y-g(X))^2] \leq E[(Y-h(X))^2]$.
   (b) Compute the correlation between $Y - g(X)$ and X^2.

3. (20 points) Let $X \sim$ Uniform$[0,1]$ and $X_n = X + I(X \in A_n)$, where $I(\cdot)$ is the indicator function, A_n is a series of intervals: $A_1 = [0,1]$, $A_2 = [0,1/2]$, $A_3 = [1/2,1]$, $A_4 = [0,1/3]$, $A_5 = [1/3,2/3]$, $A_6 = [2/3,1]$, $A_7 = [0,1/4]$, $A_8 = [1/4,2/4]$....
   (a) Does X_n converge in probability to X? Prove your answer.
   (b) Does X_n converge to a distribution? If so, what is the limit distribution? If not, explain why.
   (c) Does X_n converge almost surely to X? Why or why not?

4. (20 points) Let X_1 and X_2 be independent Gamma (ν_i, λ) random variables for $i = 1,2$. Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$.
   (a) Find the joint distribution of Y_1 and Y_2.
   (b) Find the marginal distributions of Y_1 and Y_2.

5. (20 points) Suppose that $Y_i, i = 1, \cdots, n$ are independent random variables with the following distribution:

$$\begin{aligned} Y|\lambda &\sim Pois(\lambda) \\ \log\lambda|\mu,\sigma^2 &\sim N(\mu,\sigma^2) \end{aligned} \tag{1}$$

   (a) Find the conditional mean of Y, i.e., $E(Y|\mu,\sigma^2)$.
   (b) Find the conditional variance of Y, i.e., $Var(Y|\mu,\sigma^2)$.
   (c) Find the conditions such that (1) $E(Y|\mu,\sigma^2) = Var(Y|\mu,\sigma^2)$, (2) $E(Y|\mu,\sigma^2) < Var(Y|\mu,\sigma^2)$, and (3) $E(Y|\mu,\sigma^2) > Var(Y|\mu,\sigma^2)$.
   (d) Please discuss conceptually (no mathematical deduction needed) whether the unconditional distribution of Y is still Poisson. Why or why not?

6. (20 points) Consider the joint probability density function of (U, V, W, X),

$$f(u, v, w, x) = C^{-1} I\left(u \le e^{-\frac{1}{2}x^2}\right) I\left(v \le \frac{1}{1+x^2}\right) I\left(w \le \frac{e^x}{1+e^x}\right),$$

$0 < u, v, w < 1, -\infty < x < \infty$, where C is positive constant, and $I(\cdot)$ is the indicator function with $I(a \le t) = 1$ and $I(a > t) = 0$.

(a) Write down a form for C and show that it is actually finite.

(b) Find the probability density function, $f(x)$.

(c) Show that $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ exist. In addition, show that the moment generating function of X exists.