WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics II
May, 2024

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Let $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\mu \neq 0$. Define $T_n = \frac{1}{\bar{X}_n}$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.
   (a) What is the exact variance $Var(T_n)$ at a given n?
   (b) Find an asymptotic variance of T_n.

2. (20 points) Say $X_1, \ldots, X_n$ are drawn from an autoregressive process where $X_1 \sim \mathcal{N}\left(\theta, \frac{1}{1-\rho^2}\right)$ and for $j = 2, \ldots, n$, $X_j | X_{j-1}, X_{j-2}, \ldots, X_1 \sim \mathcal{N}\left(\rho X_{j-1} + (1-\rho)\theta\right)$ where $0 \leq \rho < 1$ is a **known** constant and parameter $\theta \in \mathbb{R}$ is unknown.
   (a) Show that this distribution is from an exponential family, and find the sufficient statistic T.
   (b) Find the Cramér Rao lower bound for unbiased estimators of θ
   (c) Find the MLE for θ and show that it is unbiased

3. (20 points) Suppose $X_1, X_2, X_3, \cdots, X_n \overset{iid}{\sim} N(\mu, k^2\mu^2)$, where $k > 0$ is known and $\mu > 0$.
   (a) Find a sufficient statistics.
   (b) Is the statistic found in (a) a minimal sufficient statistics? prove or disprove your answer.
   (c) Is the statistic found in (a) a complete statistics? prove or disprove your answer.

4. (20 points) Let $X_i, i = 1, \cdots, n$ be i.i.d. $N(\mu, \sigma^2)$ random variables. Suppose that μ is known and consider estimates of σ^2 of the form,
$$S^2(k) = \frac{1}{k}\sum_{i=1}^n (X_i - \mu)^2$$
where k is a constant to be chosen. Determine the value of k which gives the smallest mean square error.

5. (20 points) For $n > 1$, let $Y_1, \ldots, Y_n$ be iid with common density
$$f_Y(y) = \sqrt{\frac{2}{\pi}}\frac{y^2}{\theta^3}\exp\left(-\frac{y^2}{2\theta^2}\right), \quad y \geq 0,$$
where $\theta > 0$ is unknown.
   (a) Find a sufficient statistic for θ.

(b) Show that $E(Y_i^2) = 3\theta^2$. Based on that, find the uniformly minimum variance unbiased estimator (UMVUE) for θ^2. [*Hint: the probability density function of gamma distribution is* $g(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}$ *for* $z > 0$, $\alpha > 0$, $\beta > 0$. *Also,* $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$.]

6. (20 points) Let $X_1, \ldots, X_n \mid \lambda \overset{ind}{\sim} \text{Poisson}(\lambda)$. Let $\bar{X}$ and S^2 denote respectively the sample mean and the sample variance, and let $T_a = a\bar{X} + (1-a)S^2, 0 \le a \le 1$.

   (a) Find a form for $\text{Std}(T_a)$ in terms of a and quantities such as $\text{Std}(S^2)$.

   (b) Deduce A and B such that $A \le \text{Std}(T_a) \le B$.

   (c) Argue that $\text{Std}(\bar{X}) \le \text{Std}(T_a)$ for all a.

   [Note that $\text{Std}(X) = +\sqrt{\text{Var}(X)}$.]