WORCESTER POLYTECHNIC INSTITUTE

THIRTY-THIRD ANNUAL INVITATIONAL MATH MEET
OCTOBER 20, 2023
INDIVIDUAL EXAM QUESTION SHEET

Directions: Please write your answers on the **INDIVIDUAL ANSWER SHEET** provided. This part of the contest is 45 minutes. Calculators and other electronics **MAY NOT** be used. Questions 1-4 are worth 1 point each, questions 5-8 are worth 2 points each, and questions 9-11 are worth 3 points each.

1 point each

1. Find a polynomial of the form $x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ with a, b, c, d, e, f integers which has $\sqrt{2} + \sqrt[3]{3}$ as a root.

   Solution: We can solve this by rearranging, cubing, rearranging, and squaring $x - (\sqrt{2} + \sqrt[3]{3}$. We get

$$\begin{aligned} x - \sqrt{2} - \sqrt[3]{3} &= 0 \\ \sqrt[3]{3} &= x - \sqrt{2} \\ 3 &= (x - \sqrt{2})^3 = x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} \\ \sqrt{2}(3x^2 + 2) &= x^3 + 6x - 3 \\ 2(3x^2 + 2)^2 &= (x^3 + 6x - 3)^2 \\ x^6 - 6x^4 - 6x^3 + 12x^2 - 36x - 1 &= 0 \end{aligned}$$

2. For real numbers x and y, simplify $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$.

   Solution: We expand out to get

$$\begin{aligned} (x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) &= x^{\frac{1}{3}}(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) - y^{\frac{1}{3}}(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) \\ &= x + x^{\frac{2}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}x^{\frac{1}{3}} - y^{\frac{1}{3}}x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\ &= x - y \end{aligned}$$

3. Compute $\log_{10}(2^{4^5}) + \log_{10}(5^{2^{10}})$.

   Solution: We compute

$$\begin{aligned} \log_{10}(2^{4^5}) + \log_{10}(5^{2^{10}}) &= 4^5 \log_{10}(2) + 2^{10} \log_{10}(5) \\ &= 1024(\log_{10}(2) + \log_{10}(5)) = 1024 \cdot \log_{10} 10 = 1024 \end{aligned}$$

4. Let $\mathbf{A}$ and $\mathbf{B}$ be vectors in $\mathbb{R}^3$. Simplify $((\mathbf{A} \times \mathbf{B}) \times \mathbf{A}) \bullet 2\mathbf{A}$.

Solution: The vector $\mathbf{P} = ((\mathbf{A} \times \mathbf{B}) \times \mathbf{A})$ is a cross product, so is perpendicular to each of the factors that we get it from, so $\mathbf{P}$ is perpendicular to $\mathbf{A} \times \mathbf{B}$, and more importantly here, $\mathbf{P}$ is perpendicular to $\mathbf{A}$. Thus $\mathbf{P} \bullet 2\mathbf{A} = 2(\mathbf{P} \bullet \mathbf{A}) = 2 \cdot 0 = 2$.

2 points each

5. Express $\frac{3}{8}$ in base 7.

   Solution: In base 7, the first term is $\frac{2}{7}$, leaving us with the problem of computing $\frac{3}{8} - \frac{2}{7} = \frac{5}{56}$ in base 7. The first term of this is $\frac{4}{49}$, leaving us with the problem of computing $\frac{5}{56} - \frac{4}{49} = \frac{3}{392}$ in base 7. The first term of this is $\frac{2}{343}$, leaving us with the problem of computing $\frac{3}{392} - \frac{2}{343} = \frac{5}{2744}$. At this point we have enough of a pattern to hope the next term is $\frac{4}{2401}$, which sure enough it is, so we get that $\left(\frac{3}{8}\right)_7 = 0.\overline{24}$.

6. Given the complex number $z = 3\sqrt{3} + 3i$, what is the smallest positive power of z which lies on the negative real axis?

   Solution: In polar, we get that $z = 3\sqrt{3} + 3i$ has magnitude $\sqrt{(3\sqrt{3})^2 + 3^2} = 6$, and angle $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$, so $z = (6, \frac{\pi}{6})$ in polar. For a power of this to lie on the negative real axis, we need $(\frac{\pi}{6})^k$ to be a multiple of π, which first happens for $k = 6$.

7. What are the solutions to $\cos(2x) = \cos(x)$ for $0 \leq x \leq \pi$?

   Solution: We use the double angle formula to get the equivalent equation $2\cos^2(x) - 1 = \cos(x)$ for $0 \leq x \leq \pi$, which we can simplify to $2\cos^2(x) - \cos(x) - 1 = 0$ and factor as $(2\cos(x) + 1)(\cos(x) - 1) = 0$, so we need $\cos(x) = 1$ or $\cos(x) = -\frac{1}{2}$. For $0 \leq x \leq \pi$, the only values that make this true are $x = 0$ and $x = \frac{2\pi}{3}$.

8. Find all real numbers x which satisfy $\left(\frac{x}{x+2}\right)^2 \geq 9$.

   Solution: We need to solve $\frac{x^2}{(x+2)^2} \geq 9$, or equivalently, $x^2 \geq 9x^2 + 36x + 36$. Subtracting we get the system $8x^2 + 36x + 36 \leq 0$, and we divide both sides by 4 to get $2x^2 + 9x + 9 \leq 0$. We can factor to get $(2x + 3)(x + 3) \leq 0$, so we need $x \geq -3$ and $x \leq -\frac{3}{2}$. This is almost correct, except we need to note our original inequality is undefined at $x = 2$, so the final answer is $[-3, -2) \cup (-2, -\frac{3}{2}]$.

3 points each

9. Consider $P(x) = (1 + x)^{1000} + x(1 + x)^{999} + x^2(1 + x)^{998} + \ldots + x^{999}(1 + x) + x^{1000}$. What is the sum of the coefficients of $P(x)$?

   Solution: We find the sum of the coefficients of a polynomial by plugging in $x = 1$. Here we get then, $P(1) = 2^{1000} + 2^{999} + 2^{998} + \ldots + 2^2 + 2 + 1$, or the sum $\sum_{i=0}^{1000} 2^i$, which equals $2^{1001} - 1$.

10. What is the square root of $99 \cdot 100 \cdot 101 \cdot 102 + 1$?

    Solution: We can do this in general. If we want the square root of $(x)(x+1)(x+2)(x+3)+1 = x^4 + 6x^3 + 11x^2 + 6x + 1$, we get that it is $(x^2 + 3x + 1)^2$, which says here that the square root of $99 \cdot 100 \cdot 101 \cdot 102 + 1$ is $99^2 + 3 \cdot 99 + 1 = 10099$.

11. Find $\cos(18^\circ)$.

Solution: We could solve this geometrically, but for purposes of writing it here, let's use our trig identities. We start by taking sin of both sides of $2 \cdot 18^\circ = 90^\circ - 3 \cdot 18^\circ$, then using our double angle formula and the relation between sin and cos to get that $2\sin(18^\circ)\cos(18^\circ) = \cos(3 \cdot 18^\circ) = 4\cos^3(18^\circ) - 3\cos(18^\circ)$, the last step coming from the triple angle formula. Rearranging, we get that $\cos(18^\circ)(2\sin(18^\circ) - 4\cos^2(18^\circ) + 3) = 0$, and since $\cos(18^\circ) \neq 0$ we can divide through by it and use $\cos^2(x) = 1 - \sin^2(x)$ to get that $4\sin^2(18^\circ) + 2\sin(18^\circ) - 1 = 0$. With the quadratic formula we get that $\sin(18^\circ) = \frac{-1+\sqrt{5}}{4}$, the positive root.

And thus we get $\cos(18^\circ) = \sqrt{1 - \sin^2(18^\circ)} = \sqrt{1 - (\frac{-1+\sqrt{5}}{4})^2} = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$.