WORCESTER POLYTECHNIC INSTITUTE

THIRTY-THIRD ANNUAL INVITATIONAL MATH MEET
OCTOBER 20, 2023
TEAM EXAM QUESTION SHEET

Directions: Please write your answers on the **TEAM ANSWER SHEET** provided. This part of the contest is 45 minutes. All 14 problems are counted equally. Calculators and other electronics **MAY NOT** be used.

1. Simplify $3333^{4444} + 4444^{3333} \mod 7$.

   Solution: We can first reduce each of the bases $\mod 7$, by noting $3333 = 7 \cdot 476 + 1$ and $4444 = 7 \cdot 634 + 6$, so we get

$$\begin{aligned} 3333^{4444} + 4444^{3333} &\equiv 1^{4444} + 6^{3333} \mod 7 \\ &\equiv 1 + (-1)^{3333} \mod 7 \\ &\equiv 1 + (-1) \mod 7 \\ &\equiv 0 \mod 7, \end{aligned}$$

   as $6 \equiv -1 \mod 7$.

2. What is the prime factorization of 1,005,010,010,005,001?

   Solution: This number has structure (it is symmetric at least) so hopefully it is some number raised to a power. We see $1000^5 = 10^{15}$ is just slightly less than this number, so let's see what we get by going up a little:

$$\begin{aligned} &1,001^1 = 1,001 \\ &1,001^2 = 1,002,001 \\ &1,001^3 = 1,002,001,000 + 1,002,001 = 1,003,003,001 \\ &1,001^4 = 1,003,003,001,000 + 1,003,003,001 = 1,004,006,004,001 \\ &1,001^5 = 1,004,006,004,001,000 + 1,004,006,004,001 = 1,005,010,010,005,001. \end{aligned}$$

   Thus we just need to factor $1,001$. We do this by inspection, trying small factors, noting 11 divides $1,001$: $1,001 = 11 \cdot 91 = 11 \cdot 13 \cdot 7$. Thus we get that

$$1,005,010,010,005,001 = (1,001)^5 = 7^5 \cdot 11^5 \cdot 13^5.$$

3. A right cylindrical cone may be constructed by taking a circular disk and removing a sector from it, matching the radii together. What angle must the removed sector have so that the resulting cone is a 45° cone, that is, so if the tip is placed at the origin, $x^2 + y^2 = z^2$?

Solution: The surface area of a cone is $\pi \cdot r \cdot l$, where r is the radius and l is the side length. Suppose the original disk had radius a; this becomes the side length of the cone. If it is a right cone, the radius of the cone r satisfies $r^2 + r^2 = a^2$, or $r = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$, so the surface area of this cone is $\frac{\sqrt{2}}{2}\pi a^2$.

For us, this should match the area of the portion of the disk which is used to make the cone, or $\pi a^2 - \frac{1}{2}\theta a^2 = (\pi - \frac{\theta}{2})a^2$. Thus $\frac{\sqrt{2}}{2}\pi = \pi - \frac{\theta}{2}$, or $\theta = 2\pi - \sqrt{2}\pi = (2-\sqrt{2})\pi$

4. Suppose S is a subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ which does not contain three integers which are pairwise relatively prime. What is the most number of elements S can have?

   Solution: 11. If it is size 12, it must contain at least three of $\{1, 2, 3, 5, 7, 11, 13\}$, as there are only 9 other options, and whichever three it contains will be relatively prime. Thus we know the answer is less than 12. We claim $S = \{2, 4, 6, 8, 10, 12, 14, 16, 3, 9, 15\}$ has the desired property and 11 elements. Any three element subset contains either two multiples of 2 or two multiples of 3, as every element falls into at least one of those two categories.

5. In $\mathbb{Z}_{11}$, integers mod 11, what is the smallest positive value of $\log_4(9)$?

   Solution: We can work out the powers of 4 in $\mathbb{Z}_{11}$ and check that $4^3 \equiv 9 \mod 11$, so the answer is 3.

6. Express $\cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x)$ in terms of a single sin or cos function.

   Solution: We know $(e^{ix})^5 = e^{i5x}$, and that $e^{ix} = \cos(x) + i\sin(x)$. Thus $\cos(5x)$ is the real part of $(\cos(x) + i\sin(x))^5$, which we compute is

$$
\begin{aligned}
\cos(5x) + i\sin(5x) =& (\cos(x) + i\sin(x))^5 \\
=& (\cos^2(x) + 2i\cos(x)\sin(x) - \sin^2(x))^2(\cos(x) + i\sin(x)) \\
=& (\cos^4(x) + 2i\cos^3(x)\sin(x) - \cos^2(x)\sin^2(x) + 2i\cos^3(x)\sin(x) \\
& - 4\cos^2(x)\sin^2(x) - 2i\cos(x)\sin^3(x) - \sin^2(x)\cos^2(x) - 2i\cos(x)\sin^3(x) \\
& + \sin^4(x))(\cos(x) + i\sin(x)) \\
=& (\cos^4(x) + 4i\cos^3(x)\sin(x) - 6\cos^2(x)\sin^2(x) - 4i\cos(x)\sin^3(x) \\
& + \sin^4(x))(\cos(x) + i\sin(x)) \\
=& \cos^5(x) + 4i\cos^4(x)\sin(x) - 6\cos^3(x)\sin^2(x) - 4i\cos^2(x)\sin^3(x) \\
& + \cos(x)\sin^4(x) + i\cos^4(x)\sin(x) - 4\cos^3(x)\sin^2(x) - 6i\cos^2(x)\sin^3(x) \\
& + 4\cos(x)\sin^4(x) + i\sin^5(x) \\
=& \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x) \\
& + i(\sin^5(x) - 10\cos^2(x)\sin^3(x) + 5\cos^4(x)\sin(x))
\end{aligned}
$$

   Thus the real part is $\cos(5x) = \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x)$. There are other, equivalent ways to write this, now that we know it is $\cos(5x)$: $\cos(5x+2n\pi)$ or $\sin(5x+\frac{\pi}{2}+2n\pi)$ for any integer n.

7. Suppose, due to wind, your flight from City A to City B travels at 500 miles per hour but on the return flight from City B to City A, the flight travels 400 miles per hour. What was your average speed over the round trip?

   Solution: Suppose the cities are d miles apart. The flight from A to B then takes $\frac{d}{500}$ hours, and the return flight takes $\frac{d}{400}$ hours, so in all, you traveled $2d$ miles in $\frac{d}{500} + \frac{d}{400} = \frac{9d}{2000}$ hours, for a rate of $\frac{2d}{\frac{9d}{2000}} = \frac{4000}{9} = 444.\bar{4}$ miles per hour on average.

8. Let $S_n = 1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n-1}n$. What is S_{2023}?

   Solution: It is easier to determine S_n for $n = 2k$, that is, n even. Suppose $n = 2k$. Then $S_n = S_{2k} = -k$. How do we know? The sum of each consecutive pair of terms is -1, so by adding $2k$ terms, we have added -1 to itself k times, for a total of $2k$. Thus $S_{2023} = S_{2022} + 2023 = -1011 + 2023 = 1012$.

9. Express 8989 as the difference of two squares of positive integers.

   Solution: If $8989 = a^2 - b^2$, then we must be able to factor 8989 as the product of two numbers $8989 = (a-b)(a+b)$. We compute $8989 = 101 \cdot 89$, so if we let $a = 95$ and $b = 6$, we get that $8989 = 89 \cdot 101 = (95-6)(95+6) = 95^2 - 6^2$. If we factored it instead as $8989 = 1 \cdot 8989$, we would get that $89894495^2 - 4494^2$.

10. Suppose $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. What is A^6?

    Solution: This is easiest if we recognize A as a rotation matrix, rotating counterclockwise by $\pi/6$. Doing this 6 times, we get rotation by π, so $A^{12} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Alternatively, we could compute matrix powers using matrix multiplication.

11. Every Pythagorean triple is associated with a right triangle. Given the triple $(12, 35, 37)$, find another triple (a, b, c) whose associated triangle has the same area (with $a \leq b < c$).

    Solution: The area of the given right triangle is $A = \frac{1}{2}bh = \frac{1}{2} \cdot 12 \cdot 35 = 210$, so we need another factorization of 210 that gives us two legs of a right triangle. Let's try $210 = 10 \cdot 21 = \frac{1}{2} \cdot 20 \cdot 21$. Is there an integer right triangle with legs of length 20 and 21? We check $\sqrt{20^2 + 21^2} = \sqrt{400 + 441} = \sqrt{841} = 29$, so one solution is $(20, 21, 29)$. This turns out to be the only factorization that works.

12. For each real number x, let $f(x) = \min(4x + 1, x + 2, -2x + 4)$. What is the maximum value of $f(x)$?

    Solution: By finding the point of intersections of these lines, we can write this function as a piecewise function: $f(x) = \begin{cases} 4x + 1 & x \leq \frac{1}{3} \\ x + 2 & \frac{1}{3} \leq x \leq \frac{2}{3} \\ -2x + 4 & \end{cases}$.

    This function is maximized when the slope changes from positive to negative, namely at the point $(x, f(x)) = (\frac{2}{3}, \frac{8}{3})$.

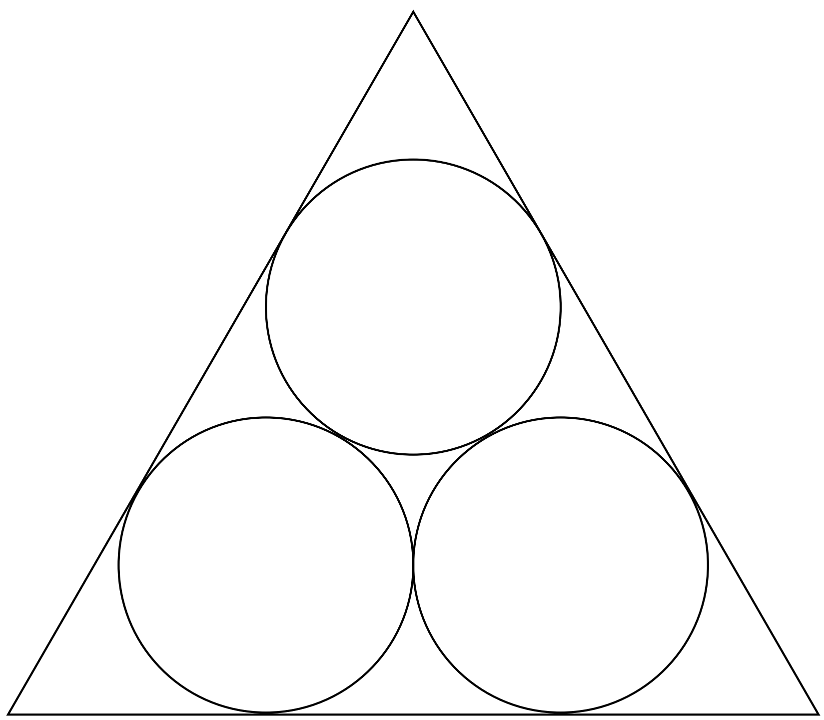
13. Suppose (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) are four collinear points on $y = x^4 + 6x^3 + 2x - 9$. Find the average of the x-coordinates, that is $\frac{x_1+x_2+x_3+x_4}{4}$.

    Solution: They are collinear, so they also lie on some line of the form $y = ax + b$ (since the vertical lines $x = a$ only intersect this curve once). Let's set these equations equal to each other and solve: we get $x^4 + 6x^3 + (2 - a)x - (9 + b) = 0$. We don't know how to find the roots of this equation, but maybe we can find $x_1 + x_2 + x_3 + x_4$. We see that we can expand out:

$$\begin{aligned}&(x - x_1)(x - x_2)(x - x_3)(x - x_4)\\ =&x^4 - (x_1 + x_2 + x_3 + x_4)x^3 + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2\\ &- (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x + x_1x_2x_3x_4\end{aligned}$$

    Thus we have that the negative of the coefficient of x^3 is the sum of the four roots, so we get that the average of the x-coordinates is $-6/4 = -\frac{3}{2}$.

14. Suppose we inscribe three circles of equal radius in an equilateral triangle of side length 3, as shown below. What is the radius of each of these three circles?



Solution: Since the triangle is equilateral, bisecting the angles gives us 30° angles. We can use this to calculate the length of the bottom (or any) side, 3, in terms of r: we get two $30^\circ - 60^\circ - 90^\circ$ right triangles giving us two factors of $\sqrt{3}r$, and then just two terms of r, for a total of $3 = (2+2\sqrt{3})r$, so $r = \frac{3}{2+2\sqrt{3}} = \frac{6-6\sqrt{3}}{-8} = \frac{3\sqrt{3}-3}{4}$.