Real Analysis GCE Name:___________________________

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Exercise 1:

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ where $f_1(x) = \sin\left(\frac{\pi}{2}x\right)$, $f_2(x) = \sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right)$, $f_3(x) = \sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right)\right)$, and so forth. Thus $f_{n+1}(x) = \sin\left(\frac{\pi}{2}f_n(x)\right)$.

1. Does this sequence converge pointwise on $[0,1]$? Prove or disprove pointwise convergence, and if the sequence converges, find the limit function f.

2. Does this sequence converge *uniformly* to f on $[0,1]$? Prove or disprove uniform convergence.

Exercise 2:

Suppose that $f \in L^1([0,\infty))$. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = f(x+n)$.

1. Show that $f_n \to 0$ in $L^1([0,1])$.

2. Show that for almost every $x \in [0,1]$, we have $\lim_{n\to\infty} f_n(x) = 0$.

Exercise 3:

Suppose $f, g \in L^\infty(0,1)$ with $f > 0$ and $g > f$. Prove or disprove that we must have $\|g\|_\infty > \|f\|_\infty$.