GCE 502 (May 2024)

Problem 1.

For an invertible matrix $A \in \mathbb{R}^{n\times n}$ and a nonzero vector $b \in \mathbb{R}^n$, consider a linear system

$$Ax = b.$$

Define $\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$, where $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$.

(1) Show that, for any solution x of the linear system,

$$\|A^{-1}\|_p \geq \frac{\|x\|_p}{\|b\|_p} \geq \|A\|_p^{-1}.$$

(2) If $p = 2$ and A is an orthogonal matrix, show that the equality holds in the above inequality.

Problem 2.

(1) Consider a $2n\times 2n$ matrix $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, where P, Q, R, S are $n\times n$ matrices. Show that, $\det A = \det P \det S$ if $R = 0$, and $\det A = (-1)^n \det Q \det R$ if $P = 0$.

(2) Suppose $B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{C}^{2\times 2}$ and $B_2 \in \mathbb{C}^{n\times n}$. Show that

$$\det B_1 \otimes B_2 = (\det B_1)^n (\det B_2)^2,$$

where $B_1 \otimes B_2$ is the Kronecker product of B_1 and B_2, i.e.

$$B_1 \otimes B_2 = \begin{bmatrix} b_{11}B_2 & b_{12}B_2 \\ b_{21}B_2 & b_{22}B_2 \end{bmatrix}$$

Problem 3.

Consider a $(n+1)\times(n+1)$ matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{bmatrix}$$

(1) Show that the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ is given by

$$p(\lambda) = \lambda^{n+1} - a_n\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda$$

(2) Show that the minimal polynomial of A is equal to the characteristic polynomial $p(\lambda)$.

Problem 4.

Let A and B be two $n \times n$ diagonalizable matrices. We say that A and B share the same eigenvectors if there exists a matrix P such that $A = PDP^{-1}$ and $B = PEP^{-1}$ for some diagonal matrices D and E.

(1) Prove that, if A and B shares the same eigenvectors, then $AB = BA$.

(2) Prove that, if $AB = BA$ and A has n distinct eigenvalues, then A and B shares the same eigenvectors.

(3) Prove or disprove the following statement: If $AB = BA$, then A and B shares the same eigenvectors.