Real Analysis GCE Name:____________________________

January, 2025

exercise 1:
Let $\{x_n\}$ be a sequence of real numbers. Show that $\{x_n\}$ converges in the extended reals if and only if $\liminf x_n = \limsup x_n$.

exercise 2:

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable with f' bounded, and $f \in L^1(\mathbb{R})$. Show that $\lim_{x\to\infty} f(x) = 0$.

exercise 3:
Suppose the sequence of functions $\{f_n\} \subset L^1([0,1])$ satisfies

$$\lim_{\lambda \to \infty} \sup_n \int_{\{x \in [0,1] : |f_n(x)| > \lambda\}} |f_n(x)| \, dx = 0.$$

Prove that $\sup_n \|f_n\|_{L^1([0,1])} < \infty$ and, in addition, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable $E \subset [0,1]$ with $|E| < \delta$, we have

$$\sup_n \int_E |f_n(x)| \, dx < \epsilon.$$