GCE - Linear Algebra

January 2025

No documents, no calculators allowed

<u>Exercise 1</u>

Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^n x = 0, T^{n-1}x \neq 0$ for some positive integer n. Show that $x, Tx, \ldots, T^{n-1}x$ are linearly independent.

<u>Exercise 2</u>

Consider the following symmetric tridiagonal matrix $n \times n$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Show that A is positive definite and all its eigenvalues are positive real numbers.

<u>Exercise 3</u>

Let A be a matrix in $\mathbb{R}^{n\times n}$. Assume that A is symmetric and positive definite, that is $x^T Ax > 0$ for all $x \neq 0$ in $\mathbb{R}^n$. Denote a_{ij}, $1 \leq i, j \leq n$, the entries of A. For $p = 1, ..., n$, let A_p be the matrix in $\mathbb{R}^{p\times p}$ defined by $A_p = (a_{ij})_{1\leq i,j\leq p}$.
Show that det $A_p > 0$.

<u>Exercise 4</u>

(i). Let A, B, C be three matrices in $K^{n\times n}$. Assume that A and B are upper-triangular and that $C = AB$. Let a_{jj}, b_{jj}, c_{jj}, $1 \leq j \leq n$, be the diagonal entries of A, B, C. Show that $c_{jj} = a_{jj} b_{jj}$.
(ii). Let M be any matrix in $\mathbb{C}^{n\times n}$. Show that

$$\det\ \exp(M) = \exp \operatorname{tr}(M),$$

where det is the determinant and tr is the trace.