WORCESTER POLYTECHNIC INSTITUTE

THIRTY-FOURTH ANNUAL INVITATIONAL MATH MEET
OCTOBER 18, 2024
INDIVIDUAL EXAM QUESTION SHEET

Directions: Please write your answers on the **INDIVIDUAL ANSWER SHEET** provided. This part of the contest is 45 minutes. Calculators and other electronics **MAY NOT** be used. Questions 1-4 are worth 1 point each, questions 5-8 are worth 2 points each, and questions 9-11 are worth 3 points each.

1 point each

1. Abby, Ben, and Charles are running around a mile-long circular track. Abby is running at 10 miles per hour, Ben at 6 miles per hour, and Charles at 4 miles per hour. If they all start running in the same direction from the starting line at noon, at what time will they all be at the starting line together again next?

   Solution: Abby is at the starting line every six minutes (ten miles per hour is one mile every six minutes), Ben is at the starting line every ten minutes, and Charles is at the starting line every fifteen minutes. We need the least common multiple of these intervals, then, and compute $\text{lcm}(6, 10, 15) = 30$, so they will all be back at the starting line together at 12:30 PM.

2. Suppose that a, b, and c are non-negative integers such that

$$\frac{23}{17} = 1 + \frac{1}{a + \frac{1}{b+\frac{1}{c}}}.$$

   Find the sum, $a + b + c$.

   Solution: Writing the improper fraction $\frac{23}{17}$ as the mixed number $1\frac{6}{17}$ tells us that $\frac{1}{a+\frac{1}{b+\frac{1}{c}}} = \frac{6}{17}$. Taking reciprocals, we get that $\frac{17}{6} = a + \frac{1}{b+\frac{1}{c}}$, and we know $\frac{17}{6} = 2\frac{5}{6}$, so $a = 2$. We get $\frac{5}{6} = \frac{1}{b+\frac{1}{c}}$, and taking reciprocals again we get $\frac{6}{5} = b + \frac{1}{c}$, so $b = 1$ and $c = 5$. Thus we get $a + b + c = 8$.

3. Find the real numbers for which $x + \frac{1}{x} > 2$.

   Solution: This is true for all $x > 0$, except $x = 1$. In this case, we can multiply through by x to get $x^2+1 > 2x$. This is true as the square of any number is positive, so $x^2-2x+1 = (x-1)^2 > 0$. If $x = 0$, the expression in the inequality is undefined, so it is not true. If $x < 0$, then when we multiply through we get that $x^2 + 1 < 2x$, which is false, again since $(x - 1)^2 > 0$. Thus this is true for x in $(0, 1) \cup (1, \infty)$.

4. Find (x, y) such that

$$\frac{4^x}{2^{x+y}} = 8 \text{ and } \frac{9^{x+y}}{3^{5y}} = 243.$$

Solution: Applying $\log_2$ to the first equation and using our logarithm rules, we get $2x - (x + y) = 3$, or $x - y = 3$. Applying $\log_3$ to the second equation and using our logarithm rules, we get $2(x + y) - 5y = 5$, or $2x - 3y = 5$. We can solve this system of equations to get $x = 4$, $y = 1$.

2 points each

5. Put these in ascending order: 2024, 2^{24}, 202^4.

Solution: We can consider each of these $\log_2$. We get that $10 < \log_2 2024 < 11$, $24 = \log_2 2^{24}$, and $28 < \log_2 202^4 = 4 \cdot \log_2 202 < 32$, so they are already given to us in order.

6. If the altitude from the right angle to the hypotenuse of a right triangle divides the hypotenuse into two segments of length $\frac{5}{2}$ and $\frac{72}{5}$, what is the length of the shortest edge of the right triangle?

Solution: The altitude to the hypotenuse splits the right triangle into two similar triangles. If we let p denote the length of the altitude, then, we get that $\frac{\frac{5}{2}}{p} = \frac{p}{\frac{72}{5}}$. Simplyifying, we compute $p = 6$. We can use the Pythagorean theorem to compute the lengths of the other two sides then. They have length $\sqrt{\left(\frac{5}{2}\right)^2 + 6^2} = \frac{13}{2}$ and $\sqrt{\left(\frac{72}{5}^2 + 6^2\right)} = \frac{78}{5}$, so the shortest side is length $\frac{13}{2}$. [We could have alternately found the side lengths by noting everything in sight is similar to a $(5, 12, 13)$ triangle.]

7. Consider the sequence $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, and $x_i = x_{i-1} - x_{i-2}$ for $i \geq 3$. Find the sum of the first 100 terms of this sequence.

Solution: Compute a few more terms of this sequence: $x_4 = -1$, $x_5 = -3$, $x_6 = -2$, $x_7 = 1$, and this sequence repeats: $x_k = x_{k-6}$ for $k \geq 7$. Further, the sum of each set of six terms is $1 + 3 + 2 + (-1) + (-3) + (-2) = 0$, so $x_1 + \ldots + x_100 = 16 \cdot 0 + x_{97} + x_{98} + x_{99} + x_{100} = x_1 + x_2 + x_3 + x_4 = 1 + 3 + 2 + (-1) = 5$.

8. Let A be the set of all positive integers which have a remainder of 1 when divided by any k with $2 \leq k \leq 10$. Find the difference between the smallest two integers in A.

Solution: If an integer is in A, it has a remainder of 1 when divided by each of $2 \leq k \leq 10$, so it is of the form $kt + 1$ for some t for each $2 \leq k \leq 11$, so it is of the form $\text{lcm}(2, 3, 4, 5, 6, 7, 8, 9, 10)t + 1$ for some t, so it is of the form $(2^3 \cdot 3^2 \cdot 5 \cdot 7)t + 1 = 2520t + 1$, so the difference between any two such consecutive terms is 2520.

3 points each

9. In the following equation, each of the letters A, B, C, and D represents uniquely a different nonzero digit in base ten. Suppose $(AB) \cdot (CB) = DDD$. Find $A + B + C + D$.

Solution: We get that $100AB + 10(AB + CB) + B^2 = DDD = D \cdot (111) = D \cdot 3 \cdot 37$. Thus D is the ones digit of B^2, which we will use later.

By prime factorization, 37 must divide one of AB or CB; let's just choose it to be AB, since in the end we are adding all four values so it doesn't matter which it is. Thus AB is 37 or 74. If $AB = 74$, then CB is one of $14, 24, 34, 54, 64, 84, 94$, so $(AB) \cdot (CB) \geq 74 \cdot 14 = 1036 > DDD$, so AB is not 74. Thus $AB = 37$. Thus D is the one's digit of $7^2 = 49$, so we get that $(37) \cdot (CB) = 999$, so $CB = 27$, so $A + B + C + D = 3 + 7 + 2 + 9 = 21$.

10. There are two different balls of two different diameters lying on the floor in two different corners of a rectangular room, each touching two walls and the floor. On each ball there is a point which is five inches from each wall that the ball touches and ten inches from the floor. Find the sum of the radii of the two balls.

    Solution: For each ball, consider the coordinate system given by the corner as the origin; then in their respective coordinate systems, each ball has the equation $(x-a)^2+(y-a)^2+(z-a)^2 = a^2$ for some a. Knowing that $(5, 5, 10)$ lies on each sphere in their respective coordinate systems, we get that $(5-a)^2 + (5-a)^2 + (10-a)^2 = a^2$, or $a^2 - 20a + 75 = 0$, so $a = 5$ or $a = 15$. These are the radii of the two spheres, so the sum of the radii of the two balls is 20 inches.

11. Find the largest possible value of k for which 3^{11} is the sum of k consecutive integers.

    Solution: We are trying to find the largest k and some nonnegative n such that $(n+1) + (n+2) + (n+3) + \ldots + (n+k) = 3^{11}$. The left hand side of this equation can be simplified to $kn + (1+2+3+\ldots+k) = kn + \frac{k(k+1)}{2}$, so equivalently, we want to find the largest k and some nonnegative n such that $2kn + k^2 + k = 2 \cdot 3^{11}$, or equivalently, $k(k+2n+1) = 2 \cdot 3^{11}$. At this point, we have a prime factorization for $k(k+2n+1)$, so we can try to maximize k and be left with a nonnegative n by checking. For example, if $k = 3^4$, then $3^4 + 2n + 1 = 2 \cdot 3^7$, implying n is still positive. If $k = 2 \cdot 3^5$, then $2 \cdot 3^5 + 2n + 1 = 3^6$, or $n = \frac{3^6 - 2 \cdot 3^5 - 1}{2}$, which is positive. If $k = 3^6$, then $n = \frac{2 \cdot 3^5 - 3^6 - 1}{2} < 0$. Thus the largest such k is $k = 2 \cdot 3^5$.