WORCESTER POLYTECHNIC INSTITUTE

THIRTY-FOURTH ANNUAL INVITATIONAL MATH MEET
OCTOBER 18, 2024
TEAM EXAM QUESTION SHEET

Directions: Please write your answers on the **TEAM ANSWER SHEET** provided. This part of the contest is 45 minutes. All 14 problems are counted equally. Calculators and other electronics **MAY NOT** be used.

1. Determine all real values of x for which

$$|x+1| + |x-3| = 4.$$

Solution: There are cases here depending on whether the absolute value functions "kicks in" or not: we need to consider $x > 3$, $-1 \le x \le 3$, and $x < -1$. If $x > 3$, then

$$|x+1| + |x-3| = x + 1 + x - 3 = 2x - 2 > 4.$$

If $x < -1$, then

$$|x+1| + |x-3| = -(x+1) - (x-3) = -2x + 2 > 4.$$

If $-1 \le x \le 3$, then

$$|x+1| + |x-3| = x + 1 - (x-3) = 4.$$

Thus the real values of x for which this is true are those in $[-1, 3]$.

2. Determine the number of quadruples of positive integers (a, b, c, d) with $a < b < c < d$ that satisfy both of the following equations:

$$ac + ad + bc + bd = 2023$$
$$a + b + c + d = 296$$

Solution: We can rewrite the first equation as $(a+b)(c+d) = 2023$ and the second as $(a+b)+(c+d) = 296$, so a change of variables might be useful: let $u = (a+b)$ and $v = (c+d)$, so we get $uv = 2023$ and $u + v = 296$. Plugging back in, we get $u(296 - u) = 2023$, or $u^2 - 296u + 2023 = 0$. We factor as $(u-7)(u-289) = 0$ to find the roots, 7 and 289. Since $a < b < c < d$, we get that $u < v$, so $u = 7$ and thus $v = 289$. Now we count quadruples, based on the choices for a and b, which must sum to 7. Thus the possible starts (a, b) to the quadruples (a, b, c, d) are $(1, 6)$, $(2, 5)$, and $(3, 4)$. In each case, we must be left with (c, d) such that $a < b < c < d$ and $c + d = 289$. Thus for $(a, b) = (1, 6)$, $7 \leq c \leq 144$, so there are 138 pairs (c, d). For $(a, b) = (2, 5)$, we get $6 \leq 144$, so there are 139 pairs (c, d). For $(a, b) = (3, 4)$, we get $5 \leq c \leq 144$, so there are 140 pairs (c, d), so there are in total $138 + 139 + 140 = 417$ such quadruples (a, b, c, d).

3. A rectangular pizza is 2 inches thick and its length is 50% longer than its width. If the entire pizza is $\frac{1}{4}$ cubic feet in volume, how many inches is its longest side?

   Solution: Let's convert the volume to cubic inches: $\frac{1}{4} * 12^3 = 432$ cubic inches. Therefore if we let w be the width of the pizza, we get $432 = V = hlw = 2 \cdot (1.5w) \cdot w = 3w^2$. Simplifying, we get that $w^2 = 144$ so $w = 12$, and $l = 18$, so the longest side is the length, at 18 inches.

4. For $0 \leq t < 2\pi$, suppose $\tan(2t) = -\frac{4}{3}$. Find the smallest positive value of $\sin(t)$.

   Solution: We use the double angle formula to get $-\frac{4}{3} = \frac{2\tan(t)}{1-\tan^2(t)}$, or simplifying, $4\tan^2(t) - 6\tan(t) - 4 = 0$, so after dividing through by 2 and factoring, we get $(2\tan(t)+1)(\tan(t)-2) = 0$, so $\tan(t) = 2$ or $\tan(t) = -\frac{1}{2}$. If $\tan(t) = 2$, then $\sin(t) = \pm\frac{2}{\sqrt{5}}$, and if $\tan(t) = -\frac{1}{2}$, then $\sin(t) = \pm\frac{1}{\sqrt{5}}$. Of these four options, the smallest positive value is $\frac{1}{\sqrt{5}}$.

5. One thousand unit cubes are fastened together to form a large cube with edge length 10. The exterior of the larger cube is painted, then this larger cube is separated into the original smaller cubes. How many of the original smaller cubes have at least one face painted?

   Solution: Each face of the large cube consists of 100 cubes; there are six of these faces. If we just add up to get 600, we are double counting the cubes on each edge of two faces and triple counting the edges in the corners. There are eight cubes on the edge between any two faces (not counting the corner cubes), and there are twelve such edges. There are six total corner cubes which are triple counted, so we get the total number of cubes with at least one painted face is $600 - 8 \times 12 - 2 \times 8 = 488$. Alternatively, the cubes which are unpainted form an $8 \times 8 \times 8$ cube inside the larger cube, so there are $10^3 - 8^3 = 488$ cubes with at least one painted face.

6. What is the smallest prime number dividing $3^{2024} + 5^{2024}$?

   Solution: Each term is odd, so their sum is even, and thus divisible by 2.

7. The interior angles of a convex polygon are an arithmetic progression beginning with $100°$ and ending with $140°$, that is, they are of the form $100°, (100 + k)°, (100 + 2k)°, \ldots, (100 + rk)° = 140°$. How many sides does the polygon have?

   Solution: Let n be the number of sides of the polygon; then in the formulation above, $r = n - 1$. The sum of the interior angles is $(n - 2) \cdot 180°$. The sum of the n terms in the arithmetic progression is the average value times n: $\frac{100+140}{2}n = 120n$. Setting these equal to each other and solving we get $n = 6$.

8. A cube of cheese $C = \{(x, y, z)$ such that $0 \leq x, y, z \leq 1\}$ is cut along the planes $x = y$, $y = z$, and $x = z$. How many pieces are there?

   Solution: The first plane $x = y$, splits the points into those with $x > y$ and those with $y < x$, and each plane does similarly with the appropriate coordinates. Thus the number of pieces of cheese is the number of possible orderings of (x, y, z), of which there are 6.

9. Suppose the sequence $\{a_i\}_{i\geq 1}$ is defined by $a_1 = 2$ and $a_{n+1} = a_n + 2n$. Find a_{2024}.

Solution: We can compute differences by rearranging the recurrence relation: $a_{n+1} - a_n = 2n$. If we add all of these from $n = 1$ to $n = 2023$, we will get $2 + 4 + 6 + \ldots + 2 \cdot (2023) = (a_{2024} - a_{2023}) + (a_{2023} - a_{2022}) + \ldots + (a_3 - a_2) + (a_2 - a_1) = a_{2024} - a_1$. We can compute this sum is two times the sum of the first 2023 integers, or $2 \cdot \frac{(2023)(2024)}{2} = 2023 \cdot 2024 = a_{2024} - a_1$, so $a_{2024} = 2023 \cdot 2024 + 2 = 4{,}094{,}554$.

10. The function f is not defined for $x = 0$, but for all non-zero real numbers x, we have $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. For which real numbers is $f(x) = f(-x)$?

Solution: We can do a change of variables $x \to \frac{1}{x}$ to yield the equation $f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$. Subtracting two of this equation from the first gives us that $-3f(x) = 3x - \frac{6}{x}$, which simplifies to $f(x) = \frac{2-x^2}{x}$. We want to know for which x $f(x) = f(-x)$, or equivalently, $\frac{2-x^2}{x} = \frac{2-(-x)^2}{-x}$. We can clear denominators to get $0 = 2x^3 - 4x = 2x(x^2 - 2)$, which has roots $x = 0$ and $x = \pm\sqrt{2}$, but $x = 0$ is not in the domain of the function. Thus the only possible values are $x = \pm\sqrt{2}$.

11. Determine the integer n that satisfies the following equation:

$$\sum_{k=81}^{99} \log_{100}\left(1 + \frac{1}{k}\right) = 2 \cdot \log_{100}\left(1 + \frac{1}{n}\right).$$

Solution: The product rule for logarithms tells us that the left hand side is equal to

$$\log_{100}\left[\left(1 + \frac{1}{81}\right)\left(1 + \frac{1}{82}\right) \cdot \ldots \cdot \left(1 + \frac{1}{99}\right)\right].$$

Rewriting each sum as an improper fraction, we get that this simplifies to

$$\begin{aligned}
\log_{100}\left[\left(1 + \frac{1}{81}\right)\left(1 + \frac{1}{82}\right) \cdot \ldots \cdot \left(1 + \frac{1}{99}\right)\right] &= \log_{100}\left[\frac{82}{81} \cdot \frac{83}{82} \cdot \ldots \cdot \frac{100}{99}\right] \\
&= \log_{100} \frac{100}{81} \\
&= \log_{100}\left(\frac{10}{9}\right)^2 \\
&= 2 \cdot \log_{100} \frac{10}{9} \\
&= 2 \cdot \log_{100}\left(1 + \frac{1}{9}\right).
\end{aligned}$$

Thus $n = 9$.

12. Factor $x^4 - 6x^2 + 1$ into a product of polynomials of lower degree, each with integer coefficients.

Solution: Using the substitution $u = x^2$, we get the quadratic $u^2 - 6u + 1$. We can't easily factor this, but can use the quadratic equation to get that $u = 3 \pm 2\sqrt{2}$, or rather, that $x^2 = 3 \pm 2\sqrt{2}$. So now we need to compute $\sqrt{3 + 2\sqrt{2}}$ and $\sqrt{3 - 2\sqrt{2}}$. Let's compute $\sqrt{3 + 2\sqrt{2}}$

first: find rational a and b such that $(a + b\sqrt{2})^2 = 3 + 2\sqrt{2}$. Simplifying, we see we need $a^2 + 2b^2 + 2ab\sqrt{2} = 3 + 2\sqrt{2}$, or rather

$$a^2 + 2b^2 = 3$$
$$2ab = 2.$$

The second equation tells us $a = \frac{1}{b}$ (since we know $b \neq 0$), which we can plug into the first to get $\frac{1}{b^2} + 2b^2 = 3$, or after clearing the denominator and collecting terms, $2b^4 - 3b^2 + 1 = 0$. Again, a change of variables for $t = b^2$ gives us the quadratic $0 = 2t^2 - 3t + 1 = (2t - 1)(t - 1)$, so $b^2 = t = 1$ or $b^2 = t = \frac{1}{2}$. In the second case, b would be irrational, so we are left just with the case where $b^2 = 1$ or $b = \pm 1$. Plugging back in, this gives us that $a = \pm 1$ as well, so the square roots of $3 + 2\sqrt{2}$ are $1 + \sqrt{2}$ and $-1 - \sqrt{2}$.

In a similar method, we can compute that the square roots of $3 - 2\sqrt{2}$ are $1 - \sqrt{2}$ and $-1 + \sqrt{2}$. Thus we can factor

$$x^4 - 6x^2 + 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})(x + 1 - \sqrt{2})(x + 1 + \sqrt{2}).$$

Our hope is that if we pair terms correctly, we will end up with two quadratics with integer coefficients, as we know now we can't factor into a linear term and a cubic. Let's just try pairing terms!

We compute

$$(x - 1 - \sqrt{2})(x - 1 + \sqrt{2}) = x^2 - x + x\sqrt{2} - x + 1 - \sqrt{2} - x\sqrt{2} + \sqrt{2} - 2 = x^2 - 2x - 1.$$

It worked! Similarly, $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2}) = x^2 + 2x - 1$, so in the end, $x^4 - 6x^2 + 1 = (x^2 - 2x - 1)(x^2 + 2x - 1)$.

Alternative solution: Once we determine the roots are irrational, we know it factors as the product of two quadratics in one of two ways: $(x^4 - 6x^2 + 1) = (x^2 + ax + 1)(x^2 + bx + 1) = x^4 + (a + b)x^3 + (2 + ab)x^2 + (a + b)x + 1$ or $(x^4 - 6x^2 + 1) = (x^2 + ax - 1)(x^2 + bx - 1) = x^4 + (a + b)x^3 + (-2 + ab)x^2 - (a + b)x + 1$. In the first case, we need $a + b = 0$ and $2 + ab = -6$ for some integers a and b. Plugging in $a = -b$, we get that $2 - b^2 = -6$, or $b^2 = 8$, or $b = \pm 2\sqrt{2}$, which is not an integer, so we must be in the second case. In this case, we need that $a + b = 0$ and $-2 + ab = -6$ for some integers a and b. Plugging in $a = -b$, we get that $-2 - b^2 = -6$, or $b^2 = 4$, or $b = \pm 2$. Thus we factor $(x^4 - 6x^2 + 1) = (x^2 - 2x - 1)(x^2 + 2x - 1)$.

13. Consider the numbers in the sequence 101, 104, 109, 116, . . ., which are of the form $a_n = 100 + n^2$ for each n in $1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n for any positive integer value of n.

    Solution: Note that d_n divides $a_n = 100 + n^2$ and $a_{n+1} = 100 + (n + 1)^2$, so it divides their difference, $2n + 1$. With this in hand, we can note that d_n divides $2(a_n) - n(2n + 1) = 200 + 2n^2 - 2n^2 - n = 200 - n$, as it divides each of the pieces. Combining these, d_n divides $2n + 1 + 2(200 - n) = 401$, so the maximum value of d_n is at most 401. Is this ever achieved? Sure enough it is, at $n = 200$. Then we can find the greatest common divisor of $100 + 200^2$ and $100 + 201^2$, as we can rewrite each of these to have a factor of 401:

$$100 + 200^2 = 100(1 + 100(2^2)) = 100 \cdot 401,$$

and

$$\begin{aligned}100 + 201^2 &= 100 + (2 \cdot 100 + 1)^2 \\ &= 100 + 4 \cdot 100^2 + 4 \cdot 100 + 1 \\ &= (100 + 1)(4 \cdot 100 + 1) = 101 \cdot 401.\end{aligned}$$

14. Consider a rectangular sheet of paper, 24 cm by 32 cm, with corners A, B, C, D in clockwise order, folded across a straight line through its center so that corners A and C now coincide. Compute the area of the resulting pentagon.

    Solution: We can see that there are two right triangles in the pentagon where the sheet is not doubled over, and we can use the dimensions of these triangles to help us find our answer. Those triangles each have one side which is 24 inches, one side which is unknown and we will call x, and a hypotenuse of $32 - x$ inches. The Pythagorean theorem tells us, then, that $x^2 + 24^2 = (32 - x)^2$, or that $64x = 448$, or that $x = 7$. This means that each triangle has an area of 84 square inches. The original piece of paper had an area of $24 \cdot 32 = 768$ square inches, so after subtracting out these two triangles, we are left with 600 square inches of area. This area has been folded over once, doubled up, so only have of it contributes to the area of the pentagon. In the end, then, the pentagon has area $300 + 84 + 84 = 468$ square inches.