WORCESTER POLYTECHNIC INSTITUTE

THIRTY-FIFTH ANNUAL INVITATIONAL MATH MEET
OCTOBER 17, 2025
INDIVIDUAL EXAM QUESTION SHEET

Directions: Please write your answers on the **INDIVIDUAL ANSWER SHEET** provided. This part of the contest is 45 minutes. Calculators and other electronics **MAY NOT** be used. Questions 1-4 are worth 1 point each, questions 5-8 are worth 2 points each, and questions 9-11 are worth 3 points each.

1 point each

1. Suppose $\log_{11}(\log_5(\log_3(x))) = 0$. Find x.

   Solution: If $\log_{11}(\log_5(\log_3(x))) = 0$, then $\log_5(\log_3(x)) = 1$, so $\log_3(x) = 5$, so $x = 3^5$.

2. Six distinct integers are chosen from $\{1, 2, 3, \ldots, 10\}$. What is the probability that 5 is the third-smallest integer picked?

   Solution: There are $\binom{10}{6} = \frac{10!}{6!4!} = 210$ total ways to choose six distinct integers from the given set. For five to be the third-smallest integer picked, we need to choose two distinct integers from $\{1, 2, 3, 4\}$ and three distinct integers from $\{6, 7, 8, 9, 10\}$; there are $\binom{4}{2}\binom{5}{3} = \frac{4!}{2!2!} \cdot \frac{5!}{3!2!} = 6 \cdot 10 = 60$ such sets. Thus the probability that 5 is the third-smallest integer picked is $\frac{60}{210} = 2/7$.

3. A closed half sphere has the property that its surface area, measured in square meters, is numerically equal to its volume, measured in cubic meters. What is its radius in meters?

   Solution: The surface area is $2\pi r^2 + \pi r^2 = 3\pi r^2$, and the volume is $\frac{2}{3}\pi r^3$. Setting them equal, we get $3r^2 = \frac{2}{3}r^3$; dividing both sides by r^2 gives $3 = \frac{2}{3}r$, or $r = 4.5$.

4. How many digits are in the base 10 representation of $5^{2025} \cdot 8^{678}$?

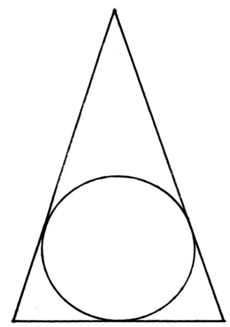
   Solution: If 8 is broken down into 2^3, the number becomes $5^{2025} \cdot (2^3)^{678} = 5^{2025} \cdot 2^{2034} = 10^{2025} \cdot 2^9 = 10^{2025} \cdot 512$, which has $2025 + 3 = 2028$ digits.

2 points each

5. Suppose $\tan(x) = a$. Find an expression for $\sin(x)\cos(x)$ as a rational function in terms of a.

   Solution: If $\tan(x) = \frac{\sin(x)}{\cos(x)} = a$ then $\sin(x) = a\cos(x)$. Recalling $\sin^2(x) + \cos^2(x) = 1$, we get $a^2\cos^2(x) + \cos^2(x) = 1$, or $\cos^2(x)(a^2 + 1) = 1$; solving, we get $\cos(x) = \frac{1}{\sqrt{a^2+1}}$. Plugging in $\cos(x) = \frac{1}{a}\sin(x)$ instead, we can similarly solve and get $\sin(x) = \frac{a}{\sqrt{a^2+1}}$. Multiplying, we get $\sin(x)\cos(x) = \frac{a}{a^2+1}$.

6. An isosceles triangle has a circle inscribed inside it as shown below, tangent to each side of the triangle. If the triangle has side lengths of a, a, and b, find an expression for the radius of the circle in terms of a and b.



Solution: Using the Pythagorean Theorem to find the height h of the triangle with b as the base, we have $(\frac{b}{2})^2 + h^2 = a^2 \implies h = \sqrt{a^2 - (\frac{b}{2})^2}$. Then the area of the whole triangle is $A = \frac{1}{2}bh = \frac{1}{2}b\sqrt{a^2 - (\frac{b}{2})^2}$. We can divide the triangle into three smaller triangles with a common point at the center of the circle, with edges emanating from the center to the vertices of the main triangle. These have areas of $\frac{1}{2}br$, $\frac{1}{2}ar$, and $\frac{1}{2}ar$. Equating their sum to the overall area and solving for r,

$$A = \frac{1}{2}br + ar = (a + \frac{1}{2}b)r \implies r = \frac{\frac{1}{2}b\sqrt{a^2 - (\frac{b}{2})^2}}{a + \frac{1}{2}b} = \frac{b}{2}\sqrt{\frac{2a-b}{2a+b}}$$

where the expression inside the square root has been factored as a difference of squares so that one factor is the same as the denominator.

7. If p, q, and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, find $p^3 + q^3 + r^3$.

   Solution: If $x^3 - x^2 + x - 2 = (x-p)(x-q)(x-r)$, then $p+q+r = 1$, $pq+qr+pr = 1$, and $pqr = 2$, by comparing like terms of x.

   We note that each is a root of $x^3 - x^2 + x - 2 = 0$, so $0 = 0+0+0 = (p^3 - p^2 + p - 2) + (q^3 - q^2 + q - 2) + (r^3 - r^2 + r - 2) = p^3 + q^3 + r^3 - (p^2 + q^2 + r^2) + (p+q+r) - 6$, or equivalently, $p^3 + q^3 + r^3 = 6 - (p+q+r) + (p^2 + q^2 + r^2)$. We know $p+q+r = 1$; we need to compute $p^2 + q^2 + r^2$. We can compute that $1 = (p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + pr)$, so $p^2 + q^2 + r^2 = 1 - 2(1) = -1$. Thus $p^3 + q^3 + r^3 = 6 - 1 - 1 = 4$.

8. For which positive integers n is $f(n) = \frac{49n-9}{n+42}$ an integer?

   Solution: We can write $49n - 9 = (n+42)(49-k)$ for some k, or $49n - 9 = 49n - kn + 2058 - 42k$; equivalently, $k(n+42) = 2067$. We can factor $2067 = 3 \cdot 689 = 3 \cdot 13 \cdot 53$. We want factorizations where $k \leq 49$, so that both terms in the factorization of $49n-9$ are positive; this corresponds to $k = 1, 3, 13, 39$, which corresponds to $n+42$ taking values of $2067, 689, 159$, and 53; accordingly, n is in $\{11, 117, 647, 2025\}$.

3 points each

9. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Compute $\sum_{n=1}^{2025} \lfloor \log_2(n) \rfloor$.

   Solution: We get that $\lfloor \log_2(n) \rfloor = k$ for $2^k \le n < 2^{k+1}$, so there are 2^k terms which contribute a value of k each. This is true for $k = 0$ through $k = 9$; for $k = 10$, there are $2025 - 2^{10} + 1 = 1002$ terms which contribute 10 each. Thus $\sum_{n=1}^{2025} \lfloor \log_2(n) \rfloor = \sum_{k=0}^{9} k \cdot 2^k + 10020$. We can either compute the first sum directly to get $0+2+8+24+64+160+384+896+2048+4608+10020 = 18214$, or use the identity that $\sum_{k=0}^{n} k \cdot 2^k = (n-1)2^{n+1} + 2$ to get to the same answer.

10. Suppose a triangle has side lengths $4\sqrt{3}$, 12, and $8\sqrt{3}$. Find the length of the angle bisector of the second largest angle.

    Solution: The second largest angle is the one opposite the second longest side; in this case, that is the side of length 12 since $1.5 < \sqrt{3} < 2$, so $4\sqrt{3} < 12 < 8\sqrt{3}$. The law of cosines, then, says that $12^2 = (4\sqrt{3})^2 + (8\sqrt{3})^2 - 2 \cdot 4\sqrt{3} \cdot 8\sqrt{3} \cos\theta$. Solving, we get that $\cos\theta = \frac{144-48-192}{-2 \cdot 4\sqrt{3} \cdot 8\sqrt{3}} = \frac{-96}{-192} = \frac{1}{2}$. Thus $\theta = 60°$. The side-angle-side formula tells us that the area of a triangle is one half the product of two sides by $\sin\theta$, where θ is the angle between the sides. In this case, if we let x be the length of the angle bisector of this $60°$ angle, we get that the area of the triangle A is $\frac{1}{2} \cdot 4\sqrt{3} \cdot 8\sqrt{3} \sin 60° = 24\sqrt{3}$. On the other hand, $A = \frac{1}{2} \cdot 4\sqrt{3}x \sin 30° + \frac{1}{2} \cdot 8\sqrt{3}x \sin 30° = 3\sqrt{3}x$. Thus $24\sqrt{3} = 3\sqrt{3}x$, so $x = 8$.

11. Find the number of distinct subsets $S \subseteq \{1, 2, \dots, 20\}$ such that the sum of elements in S leaves a remainder of 10 when divided by 32.

    Solution: Consider any subset of $\{1, 2, \dots, 20\} \setminus \{1, 2, 4, 8, 16\}$. Given such a subset, U, there is a unique subset of $\{1, 2, 4, 8, 16\}$ we should unite it with to get S, with the sum of the elements leaving a remainder of 10 when divided by 32. This is because for any number from 0 to 31, we can write it uniquely as a sum of elements in $\{1, 2, 4, 8, 16\}$; for example, $25 = 16 + 8 + 1$. So whatever the remainder when the sum of the elements in U is divided by 32 is, we can add the correct elements from $\{1, 2, 4, 8, 16\}$ to get a remainder of 10. Thus, the solution is the number of distinct subsets $U \subseteq \{1, 2, \dots, 20\} \setminus \{1, 2, 4, 8, 16\}$, of which there are $2^{15} = 32768$.