WORCESTER POLYTECHNIC INSTITUTE

THIRTY-FIFTH ANNUAL INVITATIONAL MATH MEET
OCTOBER 17, 2025
TEAM EXAM QUESTION SHEET

Directions: Please write your answers on the **TEAM ANSWER SHEET** provided. This part of the contest is 45 minutes. All 14 problems are counted equally for three points each. Calculators and other electronics **MAY NOT** be used.

1. What is $\log_9(243)$?

   Solution: It is easier to compute that $\log_3(243) = 5$ and that $3 = 9^{1/2}$. Thus $243 = 3^5 = (9^{1/2})^5 = 9^{5/2}$, so $\log_9(243) = 5/2$.

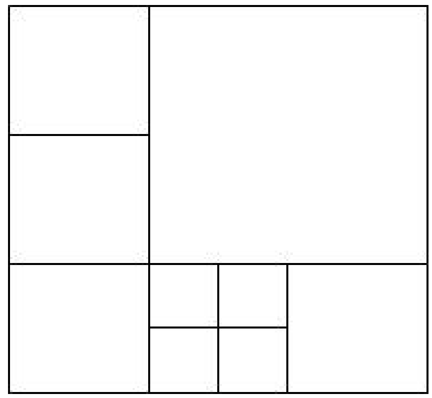
2. What is the largest positive integer m that evenly divides the number $n(n+1)(2n+1)$ for every positive integer n?

   Solution: Since n or $n+1$ is always even, this product is always divisible by 2. Also, one of the factors is always divisible by 3. If neither n nor $n+1$ is divisible by 3, then both $n-1$ and $n+2$ must be. Their sum is $2n+1$, so this number is then divisible by 3. The overall product is therefore always divisible by 6. Setting $n = 1$ gives the quantity $1 \cdot 2 \cdot 3 = 6$, showing $m = 6$ is the largest integer evenly dividing this product for all values of n.

3. Suppose $x^a x^b = 1$ and $x \neq \pm 1$. Compute $6a + 2a^2 + 6b - 2b^2 + 1$.

   Solution: Since $x^{a+b} = x^a x^b = 1$, and $x \neq \pm 1$, we have $a + b = 0$, or $a = -b$. Thus all the terms in the above expression cancel, except the constant, so $6a + 2a^2 + 6b - 2b^2 + 1 = 1$.

4. There are infinitely many ways to partition a unit square into non-overlapping smaller squares (one way is shown below). For which three positive integers k is it not possible to partition the unit square into k non-overlapping smaller squares?



Solution: If a square can be partitioned into k non-overlapping smaller squares (NOSS), then it can be partitioned into $k+3$ NOSS, by simply dividing one of the squares into 4. Further, a square can be partitioned into k NOSS for any even $k \geq 4$. How? If $k = 2k'$ is even, break the square into $k' \times k'$ smaller equal squares. In one corner, group $(k'-1) \times (k'-1)$ of these together into one big square. How many little squares are left? We compute $k' \times k' - (k' - 1) \times (k'-1) = k'^2 - (k'^2 - 2k' + 1) = 2k' - 1$ little squares, together with the big square, gives us $2k' - 1 + 1 = 2k' = k$ squares. (If $k' = 1$, this proof fails (since $k' - 1 = 0$), so this is true for $k' > 1$ or equivalently, $k \geq 4$). Thus the second statement shows that $\{4, 6, 8, 10, \ldots\}$ are all valid numbers of NOSS, and combining with the first statement shows that $\{7, 9, 11, 13, \ldots\}$ are all valid numbers of NOSS. Thus we are just left with $k = 2, 3, 5$. We don't need to check that there is no way to do this; the problem tells us there are three answers, and we have found these are the only three *possible* answers, so they are the three answers.

5. Suppose we have a rectangular prism with faces of perimeter 12, 16, and 20. Find the volume of the rectangular prism.

   Solution: Consider the side lengths of the rectangular prism, a, b, and c. Then we know $2a+2b = 12$, $2b+2c = 16$, and $2a+2c = 20$; equivalently, $a+b = 6$, $b+c = 8$, and $a+c = 10$. We need to solve this system of equations; one way to do that is to add the three equations and divide by 2 to get $a+b+c = 12$. Combining this equation with each of the above gives $\begin{cases} a = 4 \\ b = 2 \\ c = 6 \end{cases}$. Thus the volume of this rectangular prism is 48 units cubed.

6. Consider a 50×50 chessboard. A configuration of fifty pieces on the $50 \times 50 = 2500$ unit squares of this board is called *nice* if there is exactly one piece in each row and column. Find the greatest positive integer k such that for each good configuration of pieces, there is a $k \times k$ square which doesn't contain any pieces on its k^2 unit squares.

   Solution: Consider the rook in the top row; below it, place 7, aligned, 7×7 squares. The rook occupies one of these columns, so there can be at most 6 more rooks in these 7 squares, so one of them is empty. Thus $k \geq 7$. Let's show that there is a configuration with no empty 8×8 squares to show $k = 7$. Place the rooks at $(7i + j, i + 7j)$ for $\{i, j\} \in \{0, 1, 2, 3, 4, 5, 6\}$; place the last rook at $(49, 49)$. There is no 8×8 square containing no rooks, so $k < 8$. Thus $k = 7$.

7. Suppose r is a five digit, base ten number, with digits $0 \leq a, b, c, d, e \leq 9$, which is to say $r = a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e$. Then there are some values $-5 < x_1, x_2, x_3, x_4, x_5 < 5$ such that r is divisible by 13 exactly if $ax_1 + bx_2 + cx_3 + dx_4 + ex_5$ is divisible by 13. Find $x_1 + x_2 + x_3 + x_4 + x_5$.

   Solution: We get that $r = a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e \equiv 3a - b - 4c - 3d + e \pmod{13}$, computing each of the powers of 10 modulo 13, so the sum of the weights is $3 - 1 - 4 - 3 + 1 = -4$.

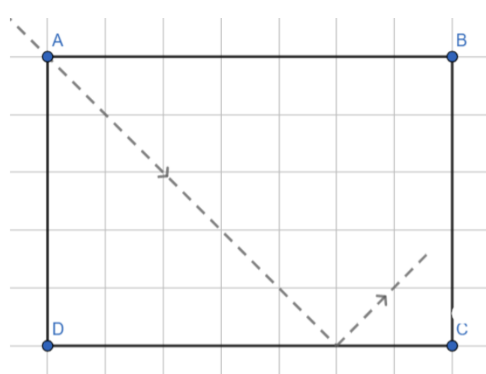
8. Consider a cube. How many ways are there to paint the faces of the cube red or blue, where two paintings are considered the same if we can rotate the cube to get from one to the other?

   Solution: Let's consider cases, based on the number of red faces. There is one way to paint a cube with 0 red faces, and one way to paint a cube with 6 red faces. All ways to paint a cube with 1 red face are equivalent under rotation; similar with 5 red faces (1 blue face). With two red faces, those faces could be opposite on the cube or not-opposite; so there are two ways to paint a cube with 2 red faces; similar with 4 red faces (2 blue faces). With 3 red faces there are two different ways to paint; either the 3 red faces share a corner, or they don't. Thus overall there are $1 + 1 + 2 + 2 + 2 + 1 + 1 = 10$ different ways to paint the cube.

9. Define the function $f(n) = n^{n^{n^{\cdots}}}$ for positive integers n where the "tower" of exponentiated n's contains exactly n n's from top to bottom, so, for example, $f(1) = 1$, $f(2) = 2^2$, and $f(6) = 6^{6^{6^{6^{6^{6}}}}}$. What is the smallest integer n such that $f(n)$ is greater than $10^{10^{100}}$?

   Solution: Let's check $f(3) < 10^{10^{100}} < f(4)$. We compute $f(3) = 3^{3^3} = 3^{27}$, which has base and exponent smaller than $10^{10^{100}}$, so is smaller. On the other hand, $4^{4^{4^4}} = 4^{4^{256}} = 4^{16^{128}} = 4^{16*16^{127}} = (4^{16})^{16^{127}}$; each level of this tower is bigger than $10^{10^{100}}$, so $f(4) > 10^{10^{100}}$.

10. There are 4 mirrors facing the inside of a 5×7 rectangle as shown in the figure. A ray of light comes into the inside of a rectangle through A with an angle of 45°. When it hits a side of the rectangle, it bounces off at the same angle, as shown in the diagram. How many times will the ray of light bounce before it reaches any one of the corners A, B, C, or D? A bounce is a time when the ray hits a mirror and reflects off it.



Solution: We can instead think of the ray as extending straight forever, and asking when it will hit one of the translates of one of the corners. If A is at the origin, then $B = (7, 0)$, $C = (0, -5)$, and $D = (7, -5)$. The translates of these points are any point of the form $(7s, -5t)$ for s, t integers. The ray hits points with equal coordinates, as it entered with an angle of $45°$, so the first translate of one of the corners it will hit is $(35, -35)$. Wrapping this back up to count bounces, the ray bounces every time it passes the horizontal line $y = -5t$ or the vertical line $x = 7s$; thus there are six horizontal bounces and four vertical bounces for a total of ten bounces.

11. How many distinct equilateral triangles can be formed by drawing three segments between vertices of a cube?

    Solution: The distance between any two vertices of a cube is either 1, if those vertices are the endpoints of a shared edge, $\sqrt{2}$ if they lie on a shared face but not a shared edge, or $\sqrt{3}$ if they do not lie on a shared face. There are no equilateral triangles of side length 1; this would correspond to a triangle of edges, of which there are none in the cube. There are no equilateral triangles of side length $\sqrt{3}$; given a vertex, it is of distance $\sqrt{3}$ from exactly one other vertex (opposite pairs), so we can't make a triangle out of them. Thus we just need to count equilateral triangles of side length $\sqrt{2}$. Starting with a fixed vertex v_0, there are three vertices of distance $\sqrt{2}$: those that share a face with v_0, which we will call v_1, v_2, and v_3. Any pair of these, along with v_0, will give us an equilateral triangle of side length $\sqrt{2}$, as none of them form opposite pairs. Thus there are $\binom{3}{2} = 3$ equilateral triangles containing any given vertex v_0, of which there are 8, but each vertex will be triple counted in the triangles, so overall there are $\frac{8 \cdot 3}{3} = 8$ such distinct triangles.

12. The square root of $8 + 2\sqrt{15}$ can be simplified to $\sqrt{x} + \sqrt{y}$ for x, y integers. Compute $x^2 + y^2$.
    Solution: We get that $8+2\sqrt{15} = 5+2\sqrt{15}+3 = (\sqrt{5}+\sqrt{3})^2$, so the square root of $8+2\sqrt{15}$ is $\sqrt{5} + \sqrt{3}$. Thus $x^2 + y^2 = 25 + 9 = 34$.

13. Find the number of lines in a three dimensional rectangular coordinate system which pass through five distinct points of the form (i, j, k) where i, j, and k are positive integers not exceeding 5.

    Solution: Suppose the first point a line passes through (i_0, j_0, k_0). Then the next points it passes through are $(i_0 + ta, j_0 + tb, k_0 + tc)$ for $t \in \{1, 2, 3, 4\}$ and for some integers a, b, c. Since the line must pass through five points all with positive integer coordinates at most 5, we get that $|a|, |b|, |c| \leq 1$; although we must rule out $a = 0 = b = c$. Thus there are $3 \times 3 \times 3 - 1 = 26$ choices of slopes of line; that said, pairs of slopes (a, b, c) and $(-a, -b, -c)$ with opposite starting points describe the same line, so there are only $\frac{26}{2} = 13$ choices of slope. For each of these, let's determine the number of possible starting points. This depends on the number of (a, b, c) which are nonzero.

    Suppose $a = 0$ and b, c are nonzero. Then j_0 and k_0 are fixed (either at 1 or 5, depending on the sign of b and c), while i_0 can take any value in $\{1, 2, 3, 4, 5\}$. There are $\frac{12}{2} = 6$ slopes with only one of (a, b, c) zero, and for each of these, then, 5 choices of starting point, for 30 total lines.

    Suppose $a = 0 = b$, and c is nonzero. Then k_0 is fixed, and i_0 and j_0 can take any value in

$\{1,2,3,4,5\}$. There are $\frac{6}{2}=3$ slopes with only one of (a,b,c) nonzero, and for each of these, 25 choices of starting point, for a total of 75 total lines.

Suppose none of the entries in (a,b,c) are zero; then i_0, j_0, and k_0 are fixed; there are $\frac{8}{2}=4$ such choices of slopes, 1 choice of starting point, for a total of 4 total lines.

Thus, in total, there are 109 total such lines.

14. Each of six bucket contains four balls. Each ball has been colored with one of n colors, such that (1) no two balls in the same bucket are the same color and (2) no two colors occur together in more than one bucket. What is the smallest possible value of n for which such a coloring is possible?

Solution: We will spend a while trying to come up with a lower bound for n, then at the end show that it is possible.

To each ball we ascribe a quantity n_b, the number of times that the color of that ball is used to color any ball. Then $\sum_{\text{all balls } b} \frac{1}{n_b} = n$; for a given color n_b, we get the term $\frac{1}{n_b}$ showing up n_b times in the sum, contributing 1 to the sum. This is the reformulation of n we will find a lower bound for: $\sum_{\text{all balls } b} \frac{1}{n_b}$. In practice, we will find a lower bound for $\sum_{\text{all balls } b \text{ in a given bucket}} \frac{1}{n_b}$, then deduce $\sum_{\text{all balls } b} \frac{1}{n_b}$ is greater than or equal to 6 times this per-bucket-bound.

Fix a bucket B, and consider $\sum_{\text{all balls } b \in B} n_b$. We want an upper bound for this term, which we then can turn into a lower bound for the sum of the reciprocals. Since no two colors occur together in more than one bucket, $\sum_{\text{all balls } b \in B} (n_b - 1) \leq 5$, the number of other buckets there are. Thus $\sum_{\text{all balls } b \in B} n_b \leq 9$.

The sum of the reciprocals will be minimized when the sum of the values are maximized, and even stronger, when the values are as close together as possible. For example, if two of these values n_{b_1} and n_{b_2} are such that $n_{b_1} \geq n_{b_2} + 2$, then $\frac{1}{n_{b_1}} + \frac{1}{n_{b_2}} > \frac{1}{n_{b_1}-1} + \frac{1}{n_{b_2}+1}$ (you can check this directly by clearing denominators!). This means that the worst case for the sum of the reciprocals is when the four values are $(2,2,2,3)$; then $\sum_{\text{all balls } b \in B} \frac{1}{n_b} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3} = \frac{11}{6}$. Putting this all together, we get

$$n = \sum_{\text{all balls } b} \frac{1}{n_b} \geq 6 \times \sum_{\text{all balls } b \text{ in a given bucket}} \frac{1}{n_b} \geq 6 \times \frac{11}{6} = 11.$$

Thus $n \geq 11$; can we do it with eleven colors? Sure, if those colors are $\{0,1,2,3,4,5,6,7,8,9,10\}$, then the buckets $B_1 = \{0,1,2,3\}, B_2 = \{0,4,5,6\}, B_3 = \{0,7,8,9\}, B_4 = \{1,4,7,10\}, B_5 = \{2,5,8,10\}, B_6 = \{3,6,9,10\}$ are such a coloring.