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1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a uniformly continuous and differentiable function. Suppose

$$\frac{f(x+h) - f(x)}{h} \to f'(x)$$

uniformly as $h \to 0$. Show that f' is uniformly continuous.

2. Show that for $C([0,1])$ (continuous functions defined on the interval $[0,1]$) with the L^∞-norm, the closed unit ball centered at zero is not compact. Hint: Find a sequence in this unit ball, with no convergent subsequence.

3. Suppose $f_n, f \colon \mathbb{R} \to \mathbb{R}$ are measurable with $0 \leq f_n \leq f$, $f_n \to f$ a.e., and $f \in L^1(\mathbb{R})$. Show that

$$\int |\sin(f_n)| \to \int |\sin(f)|.$$