GCE - Linear Algebra
January 2024

Exercise 1

Let H be a finite-dimensional real inner product space with an inner product $\langle \cdot, \cdot \rangle$.
(i). Show that if $T \in \mathcal{L}(H)$ satisfies $\langle Th, h \rangle > 0$ for all nonzero $h \in H$, then T is invertible.
(ii). If $T \in \mathcal{L}(H)$ is symmetric, show that $S = T^2 - 2T + 2I$ is invertible.
(iii). Prove or disprove: $S = T^2 - 2T + 2I$ is invertible for any $T \in \mathcal{L}(H)$.

Exercise 2

Let A be a matrix in $K^{n\times p}$, B be a matrix in $K^{p\times n}$, where $n, p \geq 1$ are two integers and K is $\mathbb{R}$ or $\mathbb{C}$.
(i). If $n = p$ and A is invertible, show that $\det(I + AB) = \det(I + BA)$.
(ii). Assume that AB and BA are diagonalizable. Let λ be a non-zero eigenvalue of AB with multiplicity r. Show that λ is an eigenvalue of BA with multiplicity r.
(iii). Assume that AB and BA are diagonalizable. Show that $\det(I + AB) = \det(I + BA)$.
(iv). Prove or disprove: if AB and BA are diagonalizable then $\det(AB) = \det(BA)$.

Exercise 3

Let V be an n-dimensional real vector space and T a linear operator on V. A subspace W in V is called as an invariant subspace (under T) if $TW \subset W$.
(i). If n is odd, show that there exists an 1-dimensional invariant subspace.
(ii). Prove or disprove: for any positive integer n, there exists an 1-dimensional invariant subspace.
(iii). Prove or disprove: for any positive integer n, there exists an invariant subspace W s.t. $\dim W \leq 2$.

Exercise 4

Let A be an invertible matrix in $\mathbb{C}^{n\times n}$. Show that there is a unitary U in $\mathbb{C}^{n\times n}$ and a positive definite matrix Hermitian matrix P in $\mathbb{C}^{n\times n}$ such that $A = UP$. **Hint:** use the matrix A^*A.