GCE - Linear Algebra
August 2026

No documents, no calculators or internet access allowed

Exercise 1

Let A, B be similar matrices, that is, $B = P^{-1}AP$.
(i). Show that $\text{trace}(A) = \text{trace}(B)$.
(ii). If P is unitary, that is, $P^* = P^{-1}$, show that $\text{trace}(A^*A) = \text{trace}(B^*B)$. Here A^* denotes the conjugate transpose of A.
(iii). Conversely, if $\text{trace}(A^*A) = \text{trace}(B^*B)$, is there a unitary matrix P such that $B = P^{-1}AP$?

Exercise 2

Let A be a 2×2 Hermitian matrix of complex numbers, that is, $A^* = A$. Let v be column vector in $\mathbb{C}^2$, and let $B = A + vv^*$. Denote the ordered eigenvalues of A and B by $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, respectively.
(i). Show that vv^* is a 2×2, positive semi-definite matrix.
(ii). Show that $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2$.
(iii). Determine the eigenvalues and eigenvectors of B if $\alpha_1 = \alpha_2$.

Exercise 3

Let A be in $\mathbb{C}^{n\times n}$. Denote $A^* = \overline{A}^T$. Recall that A is normal if, by definition, $A^*A = AA^*$.
(i). Show that if A^* is a polynomial in A, then A is normal.
(ii). Show that the converse is true. **Hint**: You may use the spectral theorem.

Exercise 4

Let $\|\|$ be the Euclidean norm on $\mathbb{R}^n$. This exercise uses the associated matrix norm on $\mathbb{R}^{n\times n}$. Let M be a non-negative symmetric matrix in $\mathbb{R}^{n\times n}$ and $0 \leq \lambda_1 \leq ... \leq \lambda_n$ the ordered eigenvalues of M.
(i). Show that λ_n maximizes $\dfrac{\|Mx\|}{\|x\|}$, for $x \neq 0$ in $\mathbb{R}^n$.
(ii). Let A be in $\mathbb{R}^{n\times n}$. Show that $\|A\|$ equals the square root of the largest eigenvalue of A^TA.
(iii). Show that

$$\lambda_{n-1} = \min_{\dim E = n-1} \max_{x \neq 0, x \in E} \frac{\|Mx\|}{\|x\|},$$

where the minimum is taken over all subspaces with dimension $n-1$.
Hint: If v_{n-1} and v_n are independent eigenvectors of A, discuss the dimension of $E \cap \text{span}\{v_{n-1}, v_n\}$.
(iv). Show that

$$\lambda_{n-1} = \min\{\|M - B\| : B \in \mathbb{R}^{n\times n}, \text{rank } B \leq 1\}.$$