General Comprehensive Exam in Real Analysis

May, 2026

1. Let $\{s_k\}$ be a sequence of real numbers, and let $\{t_n\}$ be the average value of the first n elements of $\{s_k\}$:
$$t_n = \frac{s_1 + s_2 + \ldots + s_n}{n}$$
   (a) Prove that if $\{s_k\}$ converges to a real number s, then $\{t_n\}$ also converges to s.

   (b) Give an example to show that $\{s_k\}$ may diverge, while $\{t_n\}$ still converges.

2. Suppose $f_n, f \in L^1(\mathbb{R})$ with $f_n, f \geq 0$, and for $m \in \mathbb{N}$, $x \in \mathbb{R}$ define
$$T_m(x) := \begin{cases} x & \text{if } |x| \leq m \\ m & \text{if } |x| > m. \end{cases}$$
   (a) Suppose $T_m(f_n) \to T_m(f)$ in $L^1(\mathbb{R})$ for all $m \in \mathbb{N}$. Show that this does not imply $f_n \to f$ in $L^1(\mathbb{R})$.

   (b) Suppose $T_n(f_n) - T_n(f) \to 0$ in $L^1(\mathbb{R})$. Does this imply $f_n \to f$ in $L^1(\mathbb{R})$? Justify your answer.

3. Let f be a strictly positive function defined on a measurable set $A \subset \mathbb{R}$ and $f \in L^1(A)$.

   (a) Suppose $A = [0, 1]$. Prove that
$$\inf_B \int_B f(x)\, dx > 0,$$
   where the infimum is taken over all measurable subsets $B \subset A$ with $m(B) \geq \frac{1}{2}$.

   (b) If $A = [0, \infty)$, does the same conclusion still hold?