WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 540 Probability and Mathematical Statistics I

August 2026

Note: Please show clear, logical reasoning for each solution. Even if you cannot solve a problem completely, clearly outline your approach. For example, if you run out of time before finishing a calculation, demonstrate the steps you would take to complete it and, if applicable, how you would use the result in subsequent steps.

Problem 1. Let X have a $Uniform\left(-\frac{1}{2}, \frac{1}{2}\right)$ distribution and let Z have a $Uniform(0, \frac{1}{2})$ distribution. X and Z are independent. Let $Y = X^2 + Z$.

(a) (10 points) Calculate $Cov\left(X, Y\right)$.

(b) (10 points) Are X and Y independent? Why or why not?

Problem 2. Suppose that $X_1, X_2, ..., X_n, ...$ are independent distributed such that for any n, $X_n \sim N(0, n)$. Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) (7 points) Calculate the moment generating function of Y_n.

(b) (5 points) Calculate $E(Y_n)$ and $Var(Y_n)$.

(c) (8 points) Determine the limiting distribution of Y_n using its MGF. Hint: $\sum_{a=1}^{k} a = \frac{k(k+1)}{2}$.

Problem 3. Let $0 < p < 1$ and let X and Y be independent, both with the geometric(p) distribution (i.e. $Pr(X = x) = p(1-p)^x$).

(Hint: in general, $\sum_{i=0}^{n} a^i = \frac{1-a^n}{1-a}$*; if* $|a| < 1$*,* $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$*)*

(a) What is $Pr(X = Y)$?

(b) What is $Pr(X > 2Y)$?

(c) Find the PMF for $Z = X + Y$

Problem 4. Let $Z_1, Z_2 \overset{ind}{\sim}$ Normal$(0, 1)$, and for $0 < a < 1$, let $X = aZ_1 + (1-a)Z_2$.

(a) (5 points) Use a simple argument to find the probability density function of X.

(b) (10 points) Use moment generating functions to find the probability density function of X.

(c) (5 points) Use cumulative distribution functions to find the probability density function of X.

Problem 5. Let X and Y be independent random variables with densities

$$f_X(x) = xe^{-x}, \qquad x > 0,$$

and

$$f_Y(y) = \frac{1}{2}y^2e^{-y}, \qquad y > 0.$$

Define

$$U = \frac{X}{X+Y}, \qquad V = X + Y.$$

(a) (7 points) Derive the joint density of (U, V). Then identify the marginal distributions of U and V.

(b) (6 points) Define

$$R = \frac{\max(X, Y)}{X + Y}.$$

Derive the probability density function of R.

(c) (7 points) Without evaluating a two-dimensional integral, derive an explicit expression for

$$E\left[(X - Y)^2 \mid X + Y = v\right], \qquad v > 0.$$

Justify each step using the distributional results obtained in part (a).

Problem 6. Let X be a random variable with cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}, & 0 \le x < 1, \\ \frac{1}{4} + \frac{1}{2}(x-1)^2, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

(a) (10 points) Express F_X as a mixture of a discrete cumulative distribution function F_D and a continuous cumulative distribution function F_C. That is, find a constant $p \in [0, 1]$ and cumulative distribution functions F_D and F_C such that

$$F_X(x) = pF_D(x) + (1 - p)F_C(x), \qquad -\infty < x < \infty.$$

Clearly identify p, F_D, and F_C, and justify your answer.

(b) (10 points) Let

$$Y = X^2.$$

Find the cumulative distribution function (CDF) of Y. Your final answer should be given as a piecewise-defined function. Show all necessary steps.