GCE - Linear Algebra

January 2026

No documents, no calculators or internet access allowed

Exercise 1

(i). Does there exist a 4×4 nilpotent matrix matrix N such that N^3 is not the zero matrix?
(ii). Does there exist a 4×4 nilpotent matrix matrix N such that N^4 is not the zero matrix?

Exercise 2

Let $\lambda_1, ..., \lambda_p$ be p distinct scalars in K, $K = \mathbb{R}$ or $\mathbb{C}$. Let $P(X) = (X - \lambda_1)...(X - \lambda_p)$, and for $j = 1, ..., p$, $Q_j(X) = P(X)/(X - \lambda_j)$.
(i). Show that $Q_1, ..., Q_p$ are linearly independent polynomials in the vector space $K[X]$.
(ii). Show that there are p unique scalars $c_1, ..., c_p$ in K such that $c_1 Q_1 + ... + c_p Q_p = 1$, and $c_j \neq 0$, $j = 1, ..., p$.
(iii). Let V be a vector space over K and A in $\mathcal{L}(V)$ such that $P(A) = 0$. Infer that V is the direct sum of the subspaces Ker $(A - \lambda_j I)$, $j = 1, ..., p$.

Exercise 3

Let M and B be in $\mathbb{C}^{n \times n}$.
(i). If B is diagonalizable, show that $\det(\exp(B)) = \exp(\text{tr } B)$, where tr is the trace.
(ii). Show that M is the limit of a sequence of diagonalizable matrices. **Hint:** you may use the triangular form.
(iii). Infer that the formula $\det(\exp(M)) = \exp(\text{tr } M)$ holds.

Exercise 4

Let A be in $\mathbb{C}^{n \times n}$. Denote $A^* = \overline{A}^T$. Recall that A is normal if, by definition, $A^* A = A A^*$.
(i). Show that if A^* is a polynomial in A, then A is normal.
(ii). Show that the converse is true. **Hint**: You may use the spectral theorem.