WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 541 Probability and Mathematical Statistics II

August 2026

Note: Please show clear, logical reasoning for each solution. Even if you cannot solve a problem completely, clearly outline your approach. For example, if you run out of time before finishing a calculation, demonstrate the steps you would take to complete it and, if applicable, how you would use the result in subsequent steps.

Problem 1 (20)**.** Let $X_1, X_2, ..., X_6$ be a random sample from a continuous distribution with population median m, an unknown fixed parameter. Derive a confidence interval for m with converge probability $31/32$. [Hint: Consider what the probability $P(X_{(1)} \geq m)$ means and equals to, where $X_{(1)}$ denotes the minimum among $X_1, ..., X_6$.]

Problem 2. For $i = 1, ..., n$, let $X_i \sim Bin(m, p)$, and $Y_i|X_i \sim NB(X_i + 1, 1 - p)$, meaning

$$P(Y_i = y|X_i = x) = \binom{y+x}{y}(1-p)^{(x+1)}p^y, \text{ for } y = 0, 1, 2, ..., n$$

(a) (7 points) Find a sufficient statistic for p.

(b) (7 points) Derive the uniformly most powerful test of $H_0 : p = \frac{1}{2}$ versus $H_1 : p < \frac{1}{2}$. You do not need to derive the distribution of the test statistic.

(c) (6 points) Find the maximum likelihood estimator of p.

Problem 3. Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Bernoulli}(p)$. Let $\bar{X}$ denote the sample mean and consider $\theta = p^2(1-p)^2$.

(a) (7 points) Construct an unbiased estimator of θ. Why is this estimator not optimal?

(b) (10 points) Find the best unbiased estimator (BUE) of θ. Deduce directly the expectation of the BUE. How do you know that the expectation is correct?

(c) (5 points) What can you say about the estimator, $\bar{X}^2(1-\bar{X})^2$?

Problem 4. Let $X_1, \ldots, X_n$ be *iid* with common density $f(x) = \frac{1}{\theta}x^{\frac{1}{\theta}-1}$ with $0 < x < 1$.

(a) Find the best unbiased estimator of θ

(b) Find the variance of the estimator in part (a)

Hint: is the distribution an exponential family?

Problem 5. Let $X_1, \ldots, X_n$ be iid with density

$$f(x \mid \theta) = \exp\left\{-\frac{x}{\theta} - \log\theta + \log x\right\}, \quad x > 0, \theta > 0.$$

(a) (7 points) Find a complete sufficient statistic for θ. Justify completeness.

(b) (6 points) Find the UMVUE for θ^2. Derive it explicitly and verify unbiasedness.

(c) (7 points) Let $g(\theta) = P_\theta(X_1 > 1)$. Find the UMVUE for $g(\theta)$. You may express your answer as an integral or sum.

Problem 6. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad x > 0.$$

(a) (5 points) Show that the family of distributions has a monotone likelihood ratio (MLR) in an appropriate statistic. Clearly identify the statistic.

(b) (10 points) Consider testing

$$H_0 : \theta \le \theta_0 \qquad \text{versus} \qquad H_1 : \theta > \theta_0,$$

where $\theta_0 > 0$ is known.

Derive the uniformly most powerful (UMP) level-α test. Clearly specify the rejection region. Determine the critical value so that the test has significance level α.

(c) (5 points) Derive the power function of the test.