# General Comprehensive Exam in Real Analysis

January, 2026

Note: It is not necessary to successfully complete any of the below problems in order to pass this exam. It is more important not to make false statements.

1. Let $\{f_n\}$ be a sequence of functions in $L^2([0,1])$ such that $\|f_n\|_{L^2([0,1])} < M$ for some positive constant $M$ and all $n$. Suppose $f_n \to f$ in measure for some function $f \in L^2([0,1])$. Prove that
$$\lim_{n\to\infty} \int_0^1 f_n\,\varphi\,dx = \int_0^1 f\,\varphi\,dx \qquad \text{for all } \varphi \in C([0,1]).$$

2. Suppose $\{f_n\}$ is a sequence of functions in $L^\infty([0,1])$ such that $f_n \to f$ a.e for some $f$. Suppose also that for every $x \in [0,1]$, $\exists c_x > 0$ such that $f_n(x) \geq c_x$ $\forall n$. Show that there exists $S \subset [0,1]$ with $m(S) > 0$ such that $\frac{1}{f_n} \to \frac{1}{f}$ in $L^\infty(S)$.

3. Consider real-valued functions defined on the interval $[0,1]$.

   (a) Give an example of a function that is **not** Riemann integrable on this interval, but that differs from a continuous function only on a set of Lebesgue measure zero.

   (b) Give an example of a function that is nowhere continuous on $[0,1]$, but is Lebesgue measurable on the interval.

   (c) Consider the following function on $[0,1]$. For $x$ irrational, $f(x) := 0$. For $x$ rational, write $x = \frac{p}{r}$, where $p$ and $r$ are relatively prime natural numbers, and set $f(x) := r$. Determine at which points $x_0$ the function $f$ is upper semicontinuous. $f$ is *upper semicontinuous* at $x_o$ if and only if
$$\limsup_{x\to x_o} f(x) \leq f(x_o)$$