GCE - Linear Algebra

May 2025

No documents, no calculators allowed

Exercise 1

(i). Suppose M and N are in $\mathbb{R}^{n\times n}$ such that M, N, and $M+N$ satisfy $(M+N)^{-1} = M^{-1} - N^{-1}$. Find all possible eigenvalues of MN^{-1}.
(ii). For any integer $n \geq 1$, find an example of two matrices M and N satisfying the conditions in (i).

Exercise 2

Let K be $\mathbb{R}$ or $\mathbb{C}$ and A and B two matrices in $K^{n\times n}$ such that A and B commute.
(i). Let λ be an eigenvalue of A. Show that $B(\text{Ker }(A-\lambda I)) \subset \text{Ker }(A-\lambda I)$.
(ii). Assume that A and B are diagonalizable. Show that there is a basis of K^n that diagonalizes both A and B.

Exercise 3

Let N in $\mathbb{R}^{n\times n}$ be such that $N^p = 0$, $N^{p-1} \neq 0$, for some integer $p \geq 2$. Let

$$M = \sum_{k=0}^{p-2} \frac{(-1)^k}{k+1} N^{k+1}.$$

(i). Show that $\exp(M) = I + N$.

Suggestion: for $t \in [0,1]$, set $M(t) = \displaystyle\sum_{k=0}^{p-2} \frac{(-1)^k}{k+1} t^{k+1} N^{k+1}$ and $F(t) = (I+tN)\exp(-M(t))$.
Find $F'(t)$.
(ii). For $\lambda > 0$, find a matrix A in in $\mathbb{R}^{n\times n}$ such that $\exp(A) = \lambda I + N$.

Exercise 4

Let A be an $n \times n$ Hermitian matrix with largest eigenvalue λ_1. Let B be the $(n-1)\times(n-1)$ matrix obtained by deleting the first row and first column of A. If μ_1 is the largest eigenvalue of B, prove that $\mu_1 \leq \lambda_1$.