503 General Comprehensive Exam, May 2025

1. Let $f \in L^2([0,1])$ and define $F : [0,1] \to \mathbb{R}$ by $F(x) = \int_0^x f(t)\,dt$. For any sequence $\{x_n\}$ satisfying $x_n \to 0^+$, show that

$$\lim_{n\to\infty} \frac{F(x_n)}{\sqrt{x_n}} = 0.$$

2. Suppose $f_n, f\colon \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ uniformly. Show that $\sup f_n \to \sup f$.

3. Suppose $f_n \in L^1(\mathbb{R})$ and $f_n \to f$ a.e. Show that if

$$\lim_{n\to\infty} \int f_n^2 = \int f^2 < \infty$$

and

$$\lim_{n\to\infty} \int |f_n|^{\frac{1}{2}} = \int |f|^{\frac{1}{2}} < \infty$$

then

$$\lim_{n\to\infty} \int f_n = \int f.$$