WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics I
May, 2025

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Suppose that $U_1, U_2, \ldots, U_n, U_{n+1}, \ldots$ is a sequence of iid Uniform(0,1) random variables, N has a Poisson(λ) distribution, and N is independent of U_j for all $j \geq 1$. Let $S_N = \sum_{j=1}^{N} U_j$.
   (a) Find the moment generating function of S_N.
   (b) Find $E(S_N)$ and $Var(S_N)$.

2. (20 points) Seven people (A, B, C, D, E, F, G) are randomly assigned to 7 seats (1, 2, 3, 4, 5, 6, 7) in a row. What is the probability that A and B do not seat together?

3. (20 points) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F$, where F is a continuous distribution with density f with support in the interval $[0, 1]$.

   Let m be an **odd** integer and divide the interval $[0, 1]$ in bins:

$$B_1 = \left[0, \frac{1}{m}\right), B_2 = \left[\frac{1}{m}, \frac{2}{m}\right), \ldots, B_m = \left[\frac{m-1}{m}, 1\right]$$

   Let Y_j be the number of observations that fall in B_j.

   Consider the density histogram $f_n(x) = \frac{m}{n} \sum_{j=1}^{m} Y_j I(x \in B_j)$, where $I(\cdot)$ is the indicator function.
   (a) Show that $f_n(x)$ is the density for some distribution.
   (b) For a given x, what are $\mathbb{E}\left[f_n(x)\right]$ and $Var\left(f_n(x)\right)$, in terms of $f(x)$?
   (c) Say $f(x) = 2xI(0 < x < 1)$. Show that as $n \to \infty$, $f_n\left(\frac{1}{2}\right) \xrightarrow{p} f\left(\frac{1}{2}\right) = 1$
   Hint: since m is odd, 1/2 is always the midpoint of its bin.

4. (20 points) Let $Z_s \mid \alpha_s, \beta_s \overset{ind}{\sim} \text{Beta}(\alpha_s, \beta_s), s = 1, 2$. Define $Y_1 = Z_1, Y_2 = Z_2(1 - Z_1)$.
   (a) Find the joint pdf of (Y_1, Y_2).
   (b) Find $\text{Cor}(Y_1, Y_2)$.

5. (20 points) Let X and Y be two independent standard normal variables. Let $Z = 2X - Y$ and $W = -X + Y$.
   (a) Find the joint pdf of Z and W.
   (b) Find $E(Z|W)$ and $Var(Z|W)$

6. (20 points) Let $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, with $\lambda, \mu > 0$, and assume that X and Y are independent.

Define new random variable:

$$V = \frac{X}{X+Y}$$

(a) Derive the distribution (PDF) of V.

(b) Simplify your answer in the case where $\lambda = \mu$, and provide intuition for your result.