WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics I
May, 2025

Note: Please show a clear logic of each solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} exp(\beta)$. Say you are interested in estimating $\theta = Pr(X < 1)$.
   Consider two possible estimators: (1) $\hat{\theta}_{prop}$, the proportion of observations less than 1, and (2) $\hat{\theta}_{MLE}$, the MLE. What is the asymptotic relative efficiency, $ARE\left(\hat{\theta}_{MLE}, \hat{\theta}_{prop}\right)$?

2. (20 points) Show that the MLE is asymptotic efficient. Specifically, let $X_1, X_2, \cdots, X_n$ be a sample of size n with common probability density function $f(x|\theta)$. Denote $l(\theta|x) = \sum_i \log(f(x_i|\theta))$ the log likelihood function, l' and l'' are the first and 2nd derivative of $l(\theta|x)$ w.r.t θ. Assume the MLE $\hat{\theta}$ of θ is such that $l'(\hat{\theta}|x) = 0$. Show that, as $n \to \infty$,
   $$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \frac{1}{I(\theta)}).$$
   where $I(\theta) = E[\left(\frac{\partial}{\partial\theta} \log f(X_1|\theta)\right)^2]$ is the information number. [Hint: You can use the Taylor expansion
   $$l'(\hat{\theta}|x) = l'(\theta|x) + (\hat{\theta} - \theta)l''(\theta|x) + R,$$
   where R is a negligible term (i.e., it can be ignored as $n \to \infty$). You can also use the fact that $W_i \equiv \frac{\partial}{\partial\theta} \log f(X_i|\theta) = \frac{\frac{\partial}{\partial\theta} f(X_i|\theta)}{f(X_i|\theta)}$ are i.i.d. with $E(W_i) = 0$ and $E(W_i^2) = I(\theta)$, $i = 1, \cdots, n$. Also, $U_i \equiv W_i^2 - \frac{\frac{\partial^2}{\partial\theta^2} f(X_i|\theta)}{f(X_i|\theta)}$ are i.i.d. with $E(U_i) = I(\theta)$.]

3. (20 points) Let $X_1, \ldots, X_n \mid \mu \overset{iid}{\sim} \text{Normal}(\mu, 1)$. Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$, where $\mu_1 > \mu_0$. Find the size α likelihood ratio test (LRT).

4. (20 points) Let (X_i, Y_i), $i = 1, ..., n$ be i.i.d. random vectors. Assume X_i is continuous with density $f_X(x) = x\lambda^2 \exp(-\lambda x)$ for $x > 0$. Given $X_i = x$, Y_i is normally distributed with mean θ and variance $1/x$. Assume that θ and λ are both unknown parameters.
   (a) Find a minimal sufficient statistic T. Is T complete?
   (b) Find the maximum likelihood estimators $\hat{\lambda}$ and $\hat{\theta}$ for λ and θ.
   (c) Find the marginal density of Y_i.

5. (20 points) Suppose that $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_n$, and $Z_1, Z_2, \ldots, Z_n$ are independent random samples from normal distributions with respective unknown means μ_1, μ_2, and μ_3, and common variances $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$. Suppose that we want to estimate a linear function of the means: $\theta = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$. Because the maximum-likelihood estimator (MLE) of a function of parameters is the function of the MLEs of the parameters, the MLE of θ is $\hat{\theta} = a_1\bar{X} + a_2\bar{Y} + a_3\bar{Z}$.

   (a) Find the distribution of the estimator $\hat{\theta}$. Clearly specify the mean and variance of $\hat{\theta}$.

   (b) Suppose that the sample variances of X, Y, and Z are given by S_1^2, S_2^2 and S_3^2, respectively. Give a confidence interval for θ with confidence level $1 - \alpha$.

6. (20 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with probability mass function:

$$f(x; \nu) = P(X = x) = \frac{1}{x^\nu \, \zeta(\nu)}, \quad x = 1, 2, 3, \ldots, \ \nu > 1,$$

   where the Riemann zeta function is defined by:

$$\zeta(\nu) = \sum_{x=1}^{\infty} \frac{1}{x^\nu}, \quad \text{for } \nu > 1.$$

   (a) Find a minimal sufficient statistic for the parameter ν.

   (b) Is the statistic found in part (a) complete? Prove or disprove.

   (c) Give an example of a sufficient statistic for ν that is not minimal.