General Comprehensive Exam in Real Analysis

August, 2026

1. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $[0,1]$. Suppose that there exists $\alpha > 0$ such that
$$m(E_n) \geq \alpha$$
for every $n \geq 1$. Define
$$E = \{x \in [0,1] : x \in E_n \text{ for infinitely many } n\}.$$
   (a) Prove that $m(E) \geq \alpha$.
   (b) Show by example that the conclusion in part (a) need not hold if $[0,1]$ is replaced by $\mathbb{R}$, even if $m(E_n) = 1$ for every n.

2. Suppose that $\{a_n\}$ is a Cauchy sequence of real numbers.
   (a) Show, using the definition of Cauchy sequence, that $\{a_n^2\}$ is also a Cauchy sequence.
   (b) Is the converse also true? Explain why or why not.

3. Prove or disprove: Suppose $f_n \in L^\infty(\mathbb{R})$ with $\|f_n\|_\infty \to 0$, and $g \in L^2(\mathbb{R})$. Then
$$\int_n^{n^2} f_n g \to 0.$$